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FLUCTUATIONS OF THE INCREMENT OF THE ARGUMENT FOR THE GAUSSIAN ENTIRE FUNCTION.

JEREMIAH BUCKLEY AND MIKHAIL SODIN

ABSTRACT. The Gaussian entire function is a random entire function, characterised by a certain invariance with respect to isometries of the plane. We study the fluctuations of the increment of the argument of the Gaussian entire function along planar curves. We introduce an inner product on finite formal linear combinations of curves (with real coefficients), that we call the signed length, which describes the limiting covariance of the increment. We also establish asymptotic normality of fluctuations.

Let $(\zeta_n)_{n=0}^\infty$ be a sequence of iid standard complex Gaussian random variables (that is, each ζ_n has density $\frac{1}{\pi}e^{-|z|^2}$ with respect to the Lebesgue measure on the plane), and define the Gaussian entire function by

$$(1) \quad f(z) = \sum_{n=0}^{\infty} \zeta_n \frac{z^n}{\sqrt{n!}}.$$

A remarkable feature of this random entire function is the invariance of the distribution of its zero set with respect to isometries of the plane. The invariance of the distribution of f under rotations is obvious, by the invariance of the distribution of each ζ_n . The translation invariance arises from the fact that, for any $w \in \mathbb{C}$, the Gaussian processes $f(z+w)$ and $e^{z\bar{w} + \frac{1}{2}|w|^2}f(z)$ have the same distribution; this follows, for instance, by inspecting the covariances

$$\begin{aligned} \mathbb{E} \left[e^{z_1\bar{w} + \frac{1}{2}|w|^2} f(z_1) e^{\bar{z}_2 w + \frac{1}{2}|w|^2} \overline{f(z_2)} \right] &= e^{z_1\bar{z}_2 + z_1\bar{w} + \bar{z}_2 w + |w|^2} \\ &= e^{(z_1+w)\overline{(z_2+w)}} = \mathbb{E} \left[f(z_1+w) \overline{f(z_2+w)} \right]. \end{aligned}$$

Further, by Calabi's rigidity, f is (essentially) the only Gaussian entire function whose zeroes satisfy such an invariance (see [HKPV09, Chapter 2] for details and further references).

Given a large parameter $R > 0$, the function $\log f(Rz)$ gives rise to multi-valued fields with a high intensity of logarithmic branch points, which is somewhat reminiscent of chiral bosonic fields as described by Kang and Makarov [KM13, Lecture 12]. One way to understand asymptotic fluctuations of these fields as $R \rightarrow \infty$ is to study asymptotic fluctuations of the increment of the argument of $f(Rz)$ along a given curve, which will be our concern in this paper. Note that, by the argument principle [Wik17], if the curve bounds a domain G then this observable

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coincides with the number of zeroes of f in RG (the dilation of the set G), up to a factor 2π (and a sign change if the curve is negatively oriented with respect to the domain it bounds).

We begin with the following definition.

Definition 1. In what follows a *curve* Γ is always a \mathcal{C}^1 -smooth regular oriented simple curve in the plane, of finite length¹. An \mathbb{R} -chain is a finite formal sum $\Gamma = \sum_i a_i \Gamma_i$, where Γ_i are curves and the coefficients a_i are real numbers.

Note that if the coefficients a_i are integer valued, then we can assign an obvious geometric meaning to the formal sum $\Gamma = \sum_i a_i \Gamma_i$.

Definition 2. Given a curve Γ and $R > 0$ we define $\Delta_R(\Gamma)$ to be the random variable given by the increment of the argument of $f(Rz)$ along Γ , that is,

$$\Delta_R(\Gamma) = \text{Im} \left(\int_{\Gamma} \frac{Rf'(Rz)}{f(Rz)} dz \right) = \int_{\Gamma} \frac{\partial}{\partial \hat{n}} \log |f(Rz)| |dz|$$

where \hat{n} is the unit normal to the curve Γ . Given an \mathbb{R} -chain $\Gamma = \sum_i a_i \Gamma_i$ we define $\Delta_R(\Gamma) = \sum_i a_i \Delta_R(\Gamma_i)$.

In order for this definition to make sense, we need to see that almost surely f does not vanish on a fixed curve. Note that the mean number of zeroes in a (measurable) subset of the plane is proportional to the Lebesgue measure of the set. Since the number of zeroes on a fixed curve is a non-negative random variable, whose mean is zero, the required conclusion follows. A quantitative version of this is given by [NSV08, Lemma 8].

It is worth pointing out that the observable $\Delta_R(\Gamma)$ is invariant with respect to rotations but not with respect to translations. Indeed, since the Gaussian functions $f(z+w)$ and $e^{z\bar{w} + \frac{1}{2}|w|^2} f(z)$ are equidistributed, the observable $\Delta_R(\Gamma+w)$ has the same distribution as $\Delta_R(\Gamma) + R^2 \text{Im}(\bar{w} \int_{\Gamma} dz)$. Note that the term $R^2 \text{Im}(\bar{w} \int_{\Gamma} dz)$ is not random, and that it vanishes whenever Γ is a closed chain. This implies that $\Delta_R(\Gamma+w)$ and $\Delta_R(\Gamma)$ have the same fluctuations, and furthermore hints that the mean of the random variable $\Delta_R(\Gamma)$ should be

$$(2) \quad \mathbb{E}[\Delta_R(\Gamma)] = R^2 \text{Im} \left(\int_{\Gamma} \bar{z} dz \right).$$

This formula is not difficult to justify, see the beginning of Section 2.

We are interested in studying the asymptotic fluctuations of the observable $\Delta_R(\Gamma)$, as $R \rightarrow \infty$. In order to understand the limiting covariance of $\Delta_R(\Gamma_1)$ and $\Delta_R(\Gamma_2)$ we introduce an inner product on \mathbb{R} -chains².

¹By finite length we mean finite and positive, we do not consider a single point to be a regular curve.

²Strictly speaking, we introduce an inner product on equivalence classes of \mathbb{R} -chains, where we identify two chains if their difference is the zero chain. We shall ignore this issue throughout.

Definition 3. Suppose that Γ_1 and Γ_2 are curves, whose unit normal vectors are denoted \hat{n}_1 and \hat{n}_2 respectively. We define the *signed length* of their intersection to be

$$\mathcal{L}(\Gamma_1, \Gamma_2) = \int_{\mathbb{C}} \mathbb{1}_{\Gamma_1} \mathbb{1}_{\Gamma_2} \langle \hat{n}_1, \hat{n}_2 \rangle d\mathcal{H}^1$$

where $\mathbb{1}_{\Gamma_1}$ and $\mathbb{1}_{\Gamma_2}$ are the indicator functions of the supports³ of the curves Γ_1 and Γ_2 respectively, $\langle \cdot, \cdot \rangle$ is the inner product on \mathbb{C} given by the standard inner product on \mathbb{R}^2 (we shall frequently identify \mathbb{C} with \mathbb{R}^2 without further comment) and \mathcal{H}^1 is the one-dimensional Hausdorff measure. More generally, given \mathbb{R} -chains $\Gamma_1 = \sum_i a_i \Gamma_{i,1}$ and $\Gamma_2 = \sum_j b_j \Gamma_{j,2}$ we define

$$\mathcal{L}(\Gamma_1, \Gamma_2) = \sum_{i,j} a_i b_j \mathcal{L}(\Gamma_{i,1}, \Gamma_{j,2}).$$

This definition needs several comments.

- (i) If $\gamma_k : I_k \rightarrow \mathbb{R}^2$, $k = 1, 2$, are unit speed parameterisations of the curves Γ_1 and Γ_2 then, if $\gamma_1(t_1) \in \text{image}(\gamma_2)$, we define $t_1^* = \tau(t_1) \in I_2$ to be the unique value such that $\gamma_1(t_1) = \gamma_2(t_1^*)$. We then have

$$(3) \quad \mathcal{L}(\Gamma_1, \Gamma_2) = \int_{I_1} \int_{I_2} \mathbb{1}_{\mathcal{D}}(\gamma_1(t_1), \gamma_2(t_2)) \langle \gamma_1'(t_1), \gamma_2'(t_2) \rangle d\delta_{t_1^*}(t_2) dt_1$$

where $\mathcal{D} = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 : x = y\}$ and $\delta_s(t_2)$ is the point mass at $t_2 = s$.

- (ii) Since we deal with \mathcal{C}^1 -smooth regular curves, for most of the intersection points of Γ_1 and Γ_2 the angle between the curves is either 0 or π ; there are at most countably many points where this does not hold. This means that in (3) we can replace the term $\langle \gamma_1'(t_1), \gamma_2'(t_2) \rangle$ by $\alpha(\gamma_1'(t_1), \gamma_2'(t_2))$ where

$$\alpha(x, y) = \begin{cases} +1 & \text{if } \langle x, y \rangle = |x||y|, \\ -1 & \text{if } \langle x, y \rangle = -|x||y|, \\ 0 & \text{otherwise,} \end{cases}$$

where $|\cdot|$ is the standard Euclidean norm on \mathbb{R}^2 . In other words, $\mathcal{L}(\Gamma_1, \Gamma_2)$ indeed measures the signed length of the intersection of the curves Γ_1 and Γ_2 , see Figure 1.

- (iii) The signed length is a bilinear form on \mathbb{R} -chains, that is obviously symmetric. If $\Gamma = \sum_i a_i \Gamma_i$ then the associated quadratic form is

$$\mathcal{L}(\Gamma, \Gamma) = \int_{\mathbb{R}^2} \left| \sum_i a_i \mathbb{1}_{\Gamma_i} \hat{n}_i \right|^2 d\mathcal{H}^1.$$

We see that this quadratic form is non-negative and it vanishes if and only if Γ is the zero chain, that is, $\sum_i a_i \mathbb{1}_{\Gamma_i} \hat{n}_i$ is the zero function in $L^2(\mathcal{H}^1)$. Thus the signed length defines an inner product on \mathbb{R} -chains.⁴

³By the support of a curve Γ we mean the set $\{\gamma(t) : t \in I\} \subset \mathbb{C}$ for a parameterisation $\gamma : I \rightarrow \mathbb{C}$ of Γ .

⁴It might be of some interest to describe the completion of this pre-Hilbert space, though for the purposes of this paper we shall have no need for such a description.

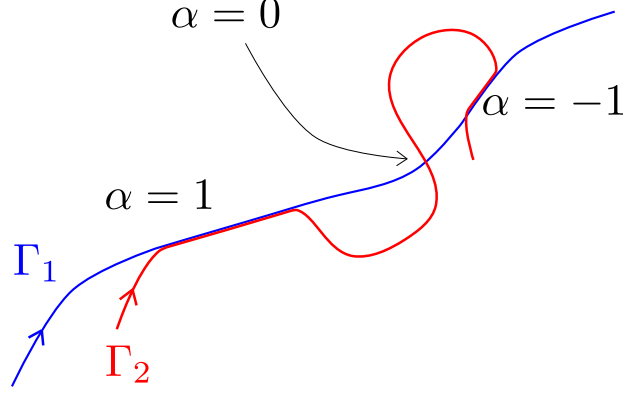


FIGURE 1. Illustration of the signed length of curves Γ_1 and Γ_2 , the value of $\alpha(\gamma'_1(t_1), \gamma'_2(t_1^*))$ is indicated at the points of intersection

We are ready to state our main result.

Theorem 1. *Let f be the Gaussian entire function (1), and let $\Gamma = \sum_i a_i \Gamma_i$ be a non-zero \mathbb{R} -chain. Then, as $R \rightarrow \infty$,*

$$(4) \quad \text{Var } \Delta_R(\Gamma) = \left(\frac{\sqrt{\pi}}{2} \zeta \left(\frac{3}{2} \right) + o(1) \right) \mathcal{L}(\Gamma, \Gamma) R,$$

where ζ is the Riemann zeta function, and the random variable

$$\frac{\Delta_R(\Gamma) - \mathbb{E}[\Delta_R(\Gamma)]}{\sqrt{\text{Var } \Delta_R(\Gamma)}}$$

converges in distribution to the standard (real) Gaussian distribution.

Less formally our result says that the observables $\Delta_R(\Gamma)$ have a scaling limit which is a Gaussian field built on the linear space of \mathbb{R} -chains equipped with the inner product defined by the signed length.

We note that the second part of the theorem can be stated as follows: Let $\Gamma_1, \dots, \Gamma_N$ be curves and define

$$\Delta_i(R) = \frac{\Delta_R(\Gamma_i) - \mathbb{E}[\Delta_R(\Gamma_i)]}{\sqrt{\text{Var } \Delta_R(\Gamma_i)}}.$$

The vector $(\Delta_1, \dots, \Delta_N)$ converges in distribution, as $R \rightarrow \infty$, to the N -dimensional Gaussian distribution with covariance Σ given by

$$\Sigma_{ij} = \frac{\mathcal{L}(\Gamma_i, \Gamma_j)}{\mathcal{L}(\Gamma_i, \Gamma_i)^{1/2} \mathcal{L}(\Gamma_j, \Gamma_j)^{1/2}}.$$

It is worth singling out a special case of Theorem 1, when each Γ_i is the positively oriented boundary of a bounded domain G_i . In this case

$$\Delta_R(\Gamma) = 2\pi \sum_i a_i n_R(G_i)$$

where $n_R(G_i)$ is the number of zeroes of the entire function f in the domain RG_i , the homothety of G_i with scaling factor R . Here the Gaussian scaling limit is built on finite linear combinations $\sum_i a_i \mathbb{1}_{G_i}$ and the limiting covariance of $n_R(G_i)$ and $n_R(G_j)$ is proportional to the signed length of $\partial G_i \cap \partial G_j$. Note that the same scaling limit (though a smaller class of domains is considered) appears in a physics paper of Lebowitz [Leb83] which deals with fluctuations of classical Coulomb systems.

The Gaussian scaling limit described in this special case corresponds to high-frequency fluctuations of linear statistics of the zero set of the Gaussian entire function f . For low frequencies the limiting Gaussian field is built on the Sobolev space W_2^2 , which consists of L^2 -functions whose weak Laplacian also belongs to L^2 . This scaling limit was described in [STs04], see also [NS11] and in particular the remark that follows [NS11, Theorem 1.1]. The co-existence of different scaling limits of linear statistics, with different scaling exponents, is a curious feature of the zeroes of the Gaussian entire function. We expect that a similar phenomenon should arise in other natural homogeneous point processes with suppressed fluctuations (so-called “superhomogeneous” or “hyperuniform” point processes, see [GL16] and the references therein).

Our work also has a one-dimensional analogue. The natural analogue of a curve in one dimension is the boundary of a finite interval and we attach a unit “normal” vector to each of the two end-points in the following manner: We say the interval is positively oriented if the normals are inward-pointing, that is, the normal on the left end-point points right, and the normal on the right end-point points left. Otherwise the interval is negatively oriented and the normals point in the opposite directions. Given two such boundaries ∂I and ∂J , denoting the respective normals \hat{n}_I and \hat{n}_J , we define an inner product by

$$\langle\langle \partial I, \partial J \rangle\rangle = \frac{1}{2} \int_{\mathbb{R}} \mathbb{1}_{\partial I} \mathbb{1}_{\partial J} \langle \hat{n}_I, \hat{n}_J \rangle d\mathcal{H}^0$$

where \mathcal{H}^0 is the (Hausdorff) counting measure, in analogy with the signed length (and we include the factor $\frac{1}{2}$ to agree exactly with the results cited below). Given an ordered pair of distinct real numbers (s, t) , we identify the pair with the boundary of an interval which is positively oriented if $s < t$ and negatively oriented if $s > t$. The corresponding inner product is then

$$\langle\langle (s, t), (s', t') \rangle\rangle = \begin{cases} 1 & \text{if } s = s' \text{ and } t = t' \\ -1 & \text{if } s = t' \text{ and } t = s' \\ \frac{1}{2} & \text{if } s = s' \text{ or } t = t' \text{ but not both} \\ -\frac{1}{2} & \text{if } s = t' \text{ or } t = s' \text{ but not both} \\ 0 & \text{otherwise.} \end{cases}$$

This inner product appears as a limiting covariance in Gaussian limit theorems for eigenvalues of random unitary matrices [DE01, Theorem 6.1; HKO01, Theorem 2.2; Wie02, Theorem 1] and the logarithm of the Riemann zeta function on the critical line [HNY08, Theorem 1 and Section 2].

We end this introduction with a brief discussion of the proof of Theorem 1. We follow the scheme developed in [STs04], which in turn goes back to [BM83]. The proof of the asymptotic

(4), after some preliminaries, boils down to Laplace-type asymptotic evaluation of certain integrals. The proof of asymptotic normality uses the method of moments, and these moments are estimated using a combinatorial argument based on the diagram method.⁵ As often happens the devil is in the details: numerous difficulties⁶ arise from the fact that we cannot say much about the intersection of two “nice” curves other than that it is a one-dimensional compact subset of the plane. For example, if $\gamma_1(t) = t$ for $0 \leq t \leq 1$ and $\gamma_2(t)$ is an arbitrary \mathcal{C}^∞ \mathbb{C} -valued function on $[0, 1]$, then the intersection of the corresponding curves can be an arbitrary closed subset of $[0, 1]$. We also mention that it seems likely that one may apply the Fourth Moment Theorem of Peccati and Tudor [PT05, Proposition 1] to see asymptotic normality, similar to [MPRW16, DNPR16]. We have not pursued this since, in our case, computing higher moments only introduces difficulties at the level of notation, and we do not think this a sufficient reason to employ such powerful machinery which relies on deep results from [NP05].

Finally, a word on notation. We write $f \lesssim g$ to mean that $f \leq Cg$ for some constant C , which may depend on certain fixed parameters. If $f \lesssim g$ and $g \lesssim f$ then we write $f \simeq g$. We write $f = O(g)$ if $|f| \lesssim g$. We write $f = o(g)$ if $\frac{f}{g} \rightarrow 0$ as $R \rightarrow \infty$. We write $f \sim g$ if $\frac{f}{g} \rightarrow 1$ as $R \rightarrow \infty$.

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1. PRELIMINARY LEMMAS

1.1. Some elementary Gaussian estimates. Suppose that ζ is a standard complex Gaussian random variable. Then a routine computation shows that for $p > -2$

$$(5) \quad \mathbb{E}[|\zeta|^p] = \Gamma(1 + \frac{p}{2}),$$

where Γ is the Euler gamma function. An immediate consequence of (5) is the following.

Lemma 2. *Let ζ be a complex Gaussian random variable and let Q be a polynomial. Then, for $1 \leq p < +\infty$,*

$$\mathbb{E}[|Q(|\zeta|^2)|^p] < +\infty.$$

The next lemma is also a simple consequence of (5).

Lemma 3. *Let ζ_1 and ζ_2 be complex Gaussian random variables with $\mathbb{E}[|\zeta_2|^2] > 0$, and let $1 \leq p < 2$. Then*

$$\mathbb{E}\left[\left|\frac{\zeta_1}{\zeta_2}\right|^p\right] < +\infty.$$

⁵We point out that there exist other approaches to asymptotic normality, based on asymptotic independence, which — given that they work in similar situations — may well apply here. See, for instance, [Tsr08, NS11, NS12] and the references therein.

⁶Note that somewhat similar difficulties were encountered by Montgomery in his study of discrepancies of uniformly distributed points [Mon94, Chapter 6, Theorem 3].

Proof. If $1 < q < \frac{2}{p}$ and q' is the Hölder conjugate of q (i.e., $\frac{1}{q} + \frac{1}{q'} = 1$), we have

$$\mathbb{E} \left[\left| \frac{\zeta_1}{\zeta_2} \right|^p \right] \leq \mathbb{E}[|\zeta_1|^{pq'}]^{\frac{1}{q'}} \mathbb{E}[|\zeta_2|^{-pq}]^{\frac{1}{q}} < +\infty. \quad \square$$

The next lemma is given as an exercise in Kahane's celebrated book, for the reader's convenience we provide a proof.

Lemma 4 ([Kah85, Chapter 12, Section 8, Exercise 3]). *Let ζ_1 and ζ_2 be jointly (complex) Gaussian random variables, with $\mathbb{E}[|\zeta_2|^2] \neq 0$. Then*

$$\mathbb{E} \left[\frac{\zeta_1}{\zeta_2} \right] = \frac{\mathbb{E}[\zeta_1 \bar{\zeta}_2]}{\mathbb{E}[|\zeta_2|^2]}.$$

Proof. Let Z_1, Z_2 be two i.i.d. $\mathcal{N}_{\mathbb{C}}(0, 1)$ random variables. Since ζ_1, ζ_2 are jointly Gaussian, there are $\alpha, \beta, \gamma \in \mathbb{C}$ such that the pair (ζ_1, ζ_2) has the same distribution as $(\alpha Z_1 + \beta Z_2, \gamma Z_1)$. In particular,

$$\frac{\zeta_1}{\zeta_2} \stackrel{d}{=} \frac{\alpha}{\gamma} + \frac{\beta}{\gamma} \frac{Z_2}{Z_1}.$$

Taking expectation, and recalling that $\mathbb{E} \left[\frac{Z_2}{Z_1} \right] = \mathbb{E}[Z_2] \mathbb{E} \left[\frac{1}{Z_1} \right] = 0$, we get

$$\mathbb{E} \left[\frac{\zeta_1}{\zeta_2} \right] = \frac{\alpha}{\gamma}.$$

All that remains is to note that $\mathbb{E}[\zeta_1 \bar{\zeta}_2] = \alpha \bar{\gamma}$ and that $\mathbb{E}[|\zeta_2|^2] = |\gamma|^2$. \square

Lemma 5 ([Fel15, Lemma B.2]). *Let ζ_1 and ζ_2 be $\mathcal{N}_{\mathbb{C}}(0, 1)$ random variables with $\mathbb{E}[\zeta_1 \bar{\zeta}_2] = \theta$ and suppose that $|\theta| \geq c > 0$ and $1 \leq p < 2$. Then*

$$\mathbb{E}[|\zeta_1 \zeta_2|^{-p}] \leq C(p, c)(1 - |\theta|^2)^{1-p}.$$

Remark. If $|\theta| = 1$ then the expectation is divergent, even for $p = 1$.

1.2. Gradients. For convenience we write $f_R(z) = f(Rz)$, $K_R(z, w) = K(Rz, Rw) = e^{R^2 z \bar{w}}$ and define

$$\widehat{f}_R(z) = \frac{f_R(z)}{\sqrt{K_R(z, z)}}$$

and note that $\widehat{f}_R(z)$ and $\widehat{f}_R(w)$ are two $\mathcal{N}_{\mathbb{C}}(0, 1)$ random variables that are jointly Gaussian and satisfy

$$\widehat{K}_R(z, w) = \mathbb{E}[\widehat{f}_R(z) \overline{\widehat{f}_R(w)}] = \frac{K_R(z, w)}{\sqrt{K_R(z, z)K_R(w, w)}}.$$

Furthermore $|\widehat{K}_R(z, w)| = e^{-R^2|z-w|^2/2}$. To simplify our notation, we define

$$g_R(z) = |\widehat{f}_R(z)|^2 = |f_R(z)|^2 e^{-R^2|z|^2}.$$

The next lemma will be important later.

Lemma 6. *Given a compact K and $1 \leq p < +\infty$, we have*

$$\mathbb{E} [|\nabla g_R(z)|^p] \leq C(p, K, R)$$

for all $z \in K$.

Proof. It is easy to see that

$$|\nabla g_R(z)| \lesssim |f'_R(z)f_R(z)|e^{-R^2|z|^2} + R^2|z||\widehat{f}_R(z)|^2.$$

Trivially $\mathbb{E} [|\widehat{f}_R(z)|^{2p}]$ is finite and independent of z , and Cauchy-Schwartz implies that

$$\mathbb{E} [|f'_R(z)f_R(z)e^{-R^2|z|^2}|^p] \leq \mathbb{E} [|f'_R(z)e^{-R^2|z|^2/2}|^{2p}]^{1/2} \mathbb{E} [|\widehat{f}_R(z)|^{2p}]^{1/2} \leq C(p, K, R),$$

since $f'_R(z)$ is a complex Gaussian with variance $(R^2 + R^4|z|^2)e^{R^2|z|^2}$. \square

Lemma 7. *Given a compact K , a polynomial Q and $1 \leq p < +\infty$, we have*

$$\mathbb{E} [|\nabla(Q \circ g_R)(z)|^p] \leq C(p, K, Q, R)$$

for all $z \in K$.

Proof. Since $\nabla(Q \circ g_R)(z) = Q'(g_R(z)) \cdot \nabla g_R(z)$ this lemma follows from Cauchy-Schwarz, Lemma 6 and Lemma 2. \square

1.3. Interchange of operations. In the proof of Theorem 1 we will repeatedly need to apply Fubini's Theorem and exchange derivatives with expectation. In this subsection we prove some lemmas that will allow us to do precisely this. Throughout this section $\Gamma_1, \dots, \Gamma_N$ will be curves and \hat{n}_j will denote the normal vector to the curve Γ_j at the point $z_j \in \Gamma_j$. We begin with a lemma that covers all of the cases we need.

Lemma 8. *Let $\psi_j : \mathbb{R}_+ \rightarrow \mathbb{R}$ be differentiable functions for $1 \leq j \leq N$ and let $\Psi_j = \psi_j \circ g_R$. Suppose that*

$$(6) \quad \int_{\prod_{j=1}^N \Gamma_j} \mathbb{E} \left[\left| \prod_{j=1}^N \nabla \Psi_j(z_j) \right| \right] \prod_{j=1}^N |dz_j| < +\infty$$

and that, for almost every tuple (z_1, \dots, z_N) with respect to the measure $\prod_{j=1}^N |dz_j|$, there exists $\varepsilon_0 > 0$ and $1 < p < 2$ such that

$$(7) \quad \sup_{w_j \in D(z_j, \varepsilon_0)} \mathbb{E} \left[\left| \prod_{j=1}^N \nabla \Psi_j(w_j) \right|^p \right] < +\infty.$$

Then

$$(8) \quad \mathbb{E} \left[\int_{\prod_{j=1}^N \Gamma_j} \frac{\partial^N}{\partial \hat{n}_1 \cdots \partial \hat{n}_N} \prod_{j=1}^N \Psi_j(z_j) \prod_{j=1}^N |dz_j| \right] \\ = \int_{\prod_{j=1}^N \Gamma_j} \frac{\partial^N}{\partial \hat{n}_1 \cdots \partial \hat{n}_N} \mathbb{E} \left[\prod_{j=1}^N \Psi_j(z_j) \right] \prod_{j=1}^N |dz_j|.$$

Remark. Trivially (6) implies that the left-hand side of (8) is well defined. However, as will be clear from the proof, we can only infer that the integrand on the right-hand side, that is the term $\frac{\partial^N}{\partial \hat{n}_1 \cdots \partial \hat{n}_N} \mathbb{E} \left[\prod_{j=1}^N \Psi_j(z_j) \right]$, is well-defined at the points where (7) holds.

Proof. Note that (6) immediately implies, by Fubini, that

$$\mathbb{E} \left[\int_{\prod_{j=1}^N \Gamma_j} \frac{\partial^N}{\partial \hat{n}_1 \cdots \partial \hat{n}_N} \prod_{j=1}^N \Psi_j(z_j) \prod_{j=1}^N |dz_j| \right] \\ = \int_{\prod_{j=1}^N \Gamma_j} \mathbb{E} \left[\frac{\partial^N}{\partial \hat{n}_1 \cdots \partial \hat{n}_N} \prod_{j=1}^N \Psi_j(z_j) \right] \prod_{j=1}^N |dz_j|.$$

It therefore suffices to show that, for almost every tuple (z_1, \dots, z_N) with respect to the measure $\prod_{j=1}^N |dz_j|$,

$$(9) \quad \mathbb{E} \left[\frac{\partial^N}{\partial \hat{n}_1 \cdots \partial \hat{n}_N} \prod_{j=1}^N \Psi_j(z_j) \right] = \frac{\partial^N}{\partial \hat{n}_1 \cdots \partial \hat{n}_N} \mathbb{E} \left[\prod_{j=1}^N \Psi_j(z_j) \right].$$

Fix a tuple (z_1, \dots, z_N) satisfying (7) for ε_0 and p , and define, for $\varepsilon_j < \varepsilon_0$,

$$h_j(\varepsilon_j) = \frac{\Psi_j(z_j + \varepsilon_j \hat{n}_j) - \Psi_j(z_j)}{\varepsilon_j}.$$

We will show that

$$(10) \quad \lim_{\varepsilon_1, \dots, \varepsilon_N \rightarrow 0} \mathbb{E} \left[\prod_{j=1}^N h_j(\varepsilon_j) \right] = \mathbb{E} \left[\lim_{\varepsilon_1, \dots, \varepsilon_N \rightarrow 0} \prod_{j=1}^N h_j(\varepsilon_j) \right]$$

which will imply (9), and therefore prove the lemma.

We begin by establishing the existence of the inner limit on the right-hand side of (10). Notice first that, almost surely, f_R does not vanish on the line intervals joining z_j to $z_j + \varepsilon_0 \hat{n}_j$. Therefore, there exist some (random) neighbourhoods of these intervals where the gradient $\nabla \Psi_j$ is a well-defined function. We conclude that the limits

$$\lim_{\varepsilon_j \rightarrow 0} h_j(\varepsilon_j) = \left\langle \nabla \Psi_j(z_j), \hat{n}_j \right\rangle = \frac{\partial}{\partial \hat{n}_j} \Psi_j(z_j)$$

exist almost surely. Finally we show that

$$(11) \quad \sup_{0 < \varepsilon_j < \varepsilon_0} \mathbb{E} \left[\left| \prod_{j=1}^N h_j(\varepsilon_j) \right|^p \right] \leq C(p).$$

By a standard argument, this implies that $\prod_{j=1}^N h_j(\varepsilon_j)$ for $0 < \varepsilon_j < \varepsilon_0$ is a uniformly integrable class of functions, and since we have already showed almost sure convergence (and therefore convergence in measure), we may infer (10).

Once more we note that, almost surely, f_R does not vanish on the line interval joining z_j to $z_j + \varepsilon_0 \hat{n}_j$. This implies that

$$|h_j(\varepsilon_j)| \leq \frac{1}{\varepsilon_j} \int_0^{\varepsilon_j} |\nabla \Psi_j(z_j + t_j \hat{n}_j)| dt_j,$$

whence,

$$|h_j(\varepsilon_j)|^p \leq \frac{1}{\varepsilon_j} \int_0^{\varepsilon_j} |\nabla \Psi_j(z_j + t_j \hat{n}_j)|^p dt_j.$$

We get

$$\begin{aligned} \mathbb{E} \left[\left| \prod_{j=1}^N h_j(\varepsilon_j) \right|^p \right] &\leq \mathbb{E} \left[\frac{1}{\varepsilon_1 \dots \varepsilon_N} \int_0^{\varepsilon_1} \dots \int_0^{\varepsilon_N} \left| \prod_{j=1}^N \nabla \Psi_j(z_j + t_j \hat{n}_j) \right|^p \prod_{j=1}^N dt_j \right] \\ &= \frac{1}{\varepsilon_1 \dots \varepsilon_N} \int_0^{\varepsilon_1} \dots \int_0^{\varepsilon_N} \mathbb{E} \left[\left| \prod_{j=1}^N \nabla \Psi_j(z_j + t_j \hat{n}_j) \right|^p \right] \prod_{j=1}^N dt_j, \end{aligned}$$

by Fubini. By (7) we see that this is bounded uniformly in $\varepsilon_1, \dots, \varepsilon_N$, which is precisely (11). \square

We now show that the hypothesis of this previous lemma hold in each of the specific cases we will need.

Lemma 9. *Suppose that ψ_j are polynomials for $1 \leq j \leq N$. Then (6) and (7) hold.*

Remark. In this case, (7) holds for every tuple (z_1, \dots, z_N) .

Proof. First note that, repeatedly applying Cauchy-Schwarz, both (6) and (7) follow if we see that

$$\mathbb{E} [|\nabla \Psi_j(z_j)|^p]$$

is uniformly bounded for z_j in a compact and any $p \geq 1$. But this is precisely the conclusion of Lemma 7. \square

Lemma 10. *Suppose that $\psi_1 = \log$ and ψ_2 is a polynomial. Then (6) and (7) hold (with $N = 2$).*

Remark. In this case, (7) holds for every pair (z_1, z_2) .

Proof. Suppose that $1 \leq p < 2$, choose $1 < q < \frac{2}{p}$ and let q' be the Hölder conjugate of q (i.e., $\frac{1}{q} + \frac{1}{q'} = 1$). Note that

$$\mathbb{E} \left[|\nabla \Psi_1(z_1) \nabla \Psi_2(z_2)|^p \right] \leq \mathbb{E} \left[|\nabla \Psi_1(z_1)|^{pq} \right]^{1/q} \mathbb{E} \left[|\nabla \Psi_2(z_2)|^{pq'} \right]^{1/q'}.$$

Once more, applying Lemma 7, the term involving Ψ_2 is uniformly bounded. It therefore suffices to see that

$$\mathbb{E} \left[|\nabla \Psi_1(z_1)|^{pq} \right]$$

is uniformly bounded for z_1 in a compact and $1 < pq < 2$. Since

$$\Psi_1(z_1) = \log |\widehat{f}_R(z_1)|^2 = \log |f_R(z_1)|^2 + R^2 |z_1|^2$$

we have

$$|\nabla \Psi_1(z_1)| \lesssim \left| \frac{f'_R(z_1)}{f_R(z_1)} \right| + R^2 |z_1|,$$

and Lemma 3 completes the proof. \square

Lemma 11. *Suppose that $\psi_1 = \psi_2 = \log$. Then (with $N = 2$) (6) holds and for every pair (z_1, z_2) with $z_1 \neq z_2$, (7) holds.*

Proof. First fix $1 \leq p < 2$, let $1 < q < \frac{2}{p}$ and let q' be the Hölder conjugate of q . Then, for $w_1 \neq w_2$,

$$\begin{aligned} \mathbb{E} \left[\left| \frac{f'_R(w_1)}{f_R(w_1)} \frac{f'_R(w_2)}{f_R(w_2)} \right|^p \right] &\leq \left(\mathbb{E} \left[|f_R(w_1) f_R(w_2)|^{-pq} \right] \right)^{\frac{1}{q}} \left(\mathbb{E} \left[|f'_R(w_1) f'_R(w_2)|^{pq'} \right] \right)^{\frac{1}{q'}} \\ &\leq \left(\mathbb{E} \left[|f_R(w_1) f_R(w_2)|^{-pq} \right] \right)^{\frac{1}{q}} \left(\mathbb{E} \left[|f'_R(w_1)|^{2pq'} \right] \mathbb{E} \left[|f'_R(w_2)|^{2pq'} \right] \right)^{\frac{1}{2q'}} \\ &\leq C \left(\mathbb{E} \left[|f_R(w_1) f_R(w_2)|^{-pq} \right] \right)^{\frac{1}{q}} \end{aligned}$$

since $f'_R(w)$ is a complex Gaussian with variance $(R^2 + R^4 |w|^2) e^{R^2 |w|^2}$. (The constant C depends on p, R and, if w_1 and w_2 are restricted to lie in a compact K , on K .) Applying Lemma 5 we have

$$(12) \quad \mathbb{E} \left[\left| \frac{f'_R(w_1)}{f_R(w_1)} \frac{f'_R(w_2)}{f_R(w_2)} \right|^p \right] \lesssim (1 - |\widehat{K}_R(w_1, w_2)|^2)^{\frac{1}{q} - p}.$$

Once more we note that $|\nabla \Psi_j(z)| \lesssim \left| \frac{f'_R(z)}{f_R(z)} \right| + R^2 |z|$ for $j = 1, 2$. Therefore to show (6) it suffices to see that

$$\int_{\Gamma_1} \int_{\Gamma_2} \mathbb{E} \left[\left| \frac{f'_R(z_1)}{f_R(z_1)} \frac{f'_R(z_2)}{f_R(z_2)} \right| \right] |dz_1| |dz_2| \leq C.$$

Now for $z_1 \in \Gamma_1$ and $z_2 \in \Gamma_2$ we have $1 - |\widehat{K}_R(z_1, z_2)|^2 = 1 - e^{-R^2|z_1 - z_2|^2} \gtrsim R^2|z_1 - z_2|^2$, where the implicit constant depends only on Γ_1 and Γ_2 . We conclude that, taking $p = 1$ in (12),

$$\int_{\Gamma_2} \int_{\Gamma_1} \mathbb{E} \left[\left| \frac{f'_R(z_1)}{f_R(z_1)} \frac{f'_R(z_2)}{f_R(z_2)} \right| \right] |dz_1| |dz_2| \lesssim \int_{\Gamma_2} \left(\int_{\Gamma_1 \setminus \{z_2\}} |z_1 - z_2|^{2(\frac{1}{q}-1)} |dz_1| \right) |dz_2| < +\infty,$$

since $2(\frac{1}{q} - 1) > -1$. This proves (6).

Now fix $z_1 \in \Gamma_1$ and $z_2 \in \Gamma_2$ with $z_1 \neq z_2$ and $1 < p < 2$. Since $|\widehat{K}_R(z_1, z_2)| = e^{-R^2|z_1 - z_2|^2/2} < 1$ we can find $\varepsilon_0 > 0$ such that

$$\sup_{\substack{w_1 \in D(z_1, \varepsilon_0) \\ w_2 \in D(z_2, \varepsilon_0)}} |\widehat{K}_R(w_1, w_2)| < 1.$$

Then, by (12),

$$\sup_{\substack{w_1 \in D(z_1, \varepsilon_0) \\ w_2 \in D(z_2, \varepsilon_0)}} \mathbb{E} \left[\left| \frac{f'_R(w_1)}{f_R(w_1)} \frac{f'_R(w_2)}{f_R(w_2)} \right|^p \right] \lesssim \sup_{\substack{w_1 \in D(z_1, \varepsilon_0) \\ w_2 \in D(z_2, \varepsilon_0)}} (1 - |\widehat{K}_R(w_1, w_2)|^2)^{\frac{1}{q}-p} < +\infty.$$

This implies that (7) holds, and completes the proof of the lemma. \square

2. THE MEAN AND VARIANCE

In this section we prove the first part of our theorem, the asymptotic (4). We begin by computing the mean of $\Delta_R(\Gamma)$, that is, proving (2); note that by linearity that it's enough to show that

$$\mathbb{E}[\Delta_R(\Gamma)] = R^2 \operatorname{Im} \left(\int_{\Gamma} \bar{z} dz \right)$$

for a \mathcal{C}^1 regular oriented simple curve Γ . For such a curve we recall that

$$\Delta_R(\Gamma) = \operatorname{Im} \left(\int_{\Gamma} \frac{f'_R(z)}{f_R(z)} dz \right)$$

which implies that

$$\mathbb{E}[\Delta_R(\Gamma)] = \operatorname{Im} \left(\int_{\Gamma} \mathbb{E} \left[\frac{f'_R(z)}{f_R(z)} \right] dz \right);$$

we may apply Fubini by Lemma 3. Applying Lemma 4 we see that

$$\mathbb{E}[\Delta_R(\Gamma)] = \operatorname{Im} \left(\int_{\Gamma} \frac{R^2 \bar{z} e^{R^2|z|^2}}{e^{R^2|z|^2}} dz \right) = R^2 \operatorname{Im} \left(\int_{\Gamma} \bar{z} dz \right),$$

which is precisely (2).

2.1. **The variance.** Given a chain $\Gamma = \sum_i a_i \Gamma_i$, to prove (4) it is enough to show that

$$(13) \quad \text{Cov}(\Delta_R(\Gamma_i), \Delta_R(\Gamma_j)) = \frac{\sqrt{\pi}}{2} \zeta\left(\frac{3}{2}\right) R\mathcal{L}(\Gamma_i, \Gamma_j)(1 + o(1))$$

and the rest of this section will be devoted to establishing this asymptotic. First recall that

$$(14) \quad \Delta_R(\Gamma_i) = \int_{\Gamma_i} \frac{\partial}{\partial \hat{n}_i} \log |f_R(z)| |dz|$$

and note that (2) may be re-written as

$$(15) \quad \mathbb{E}[\Delta_R(\Gamma_i)] = \frac{1}{2} \int_{\Gamma_i} \frac{\partial}{\partial \hat{n}_i} \log K_R(z, z) |dz|.$$

Recalling that

$$\hat{f}_R(z) = \frac{f_R(z)}{\sqrt{K_R(z, z)}},$$

we see that

$$\text{Cov}(\Delta_R(\Gamma_i), \Delta_R(\Gamma_j)) = \mathbb{E} \left[\int_{\Gamma_i} \int_{\Gamma_j} \frac{\partial^2}{\partial \hat{n}_i \partial \hat{n}_j} \log |\hat{f}_R(z_j)| \log |\hat{f}_R(z_i)| |dz_j| |dz_i| \right].$$

Note that here, and henceforth unless specified otherwise, \hat{n}_i (respectively \hat{n}_j) refers to the unit normal vector to the curve Γ_i (respectively Γ_j) at the point $z_i \in \Gamma_i$ (respectively $z_j \in \Gamma_j$).

Now Lemma 11 allows us to apply Lemma 8 to see that

$$(16) \quad \text{Cov}(\Delta_R(\Gamma_i), \Delta_R(\Gamma_j)) = \int_{\Gamma_i} \int_{\Gamma_j} \frac{\partial^2}{\partial \hat{n}_i \partial \hat{n}_j} \mathbb{E} \left[\log |\hat{f}_R(z_j)| \log |\hat{f}_R(z_i)| \right] |dz_j| |dz_i|.$$

We add the caveat here (c.f. the remark to Lemma 8) that the integrand on the right-hand side is defined only for $z_j \neq z_i$. We compute the inner expectation through the following lemma.

Lemma 12 ([SZ08, Lemma 3.3; NS11, Lemma 2.2; HKPV09, Lemma 3.5.2]). *If ζ_1 and ζ_2 are $\mathcal{N}_{\mathbb{C}}(0, 1)$ random variables with $\mathbb{E}[\zeta_1 \bar{\zeta}_2] = \theta$ then*

$$\mathbb{E}[\log |\zeta_1|] = -\frac{\gamma}{2}$$

and

$$\text{Cov}[\log |\zeta_1|, \log |\zeta_2|] = \frac{1}{4} \text{Li}_2(|\theta|^2)$$

where the dilogarithm is defined by

$$\text{Li}_2(z) = \sum_{\alpha=1}^{\infty} \frac{z^\alpha}{\alpha^2}.$$

Applying the lemma and recalling that

$$|\hat{K}_R(z_j, z_i)| = \frac{|K_R(z_j, z_i)|}{\sqrt{K_R(z_j, z_j) K_R(z_i, z_i)}} = e^{-R^2 |z_j - z_i|^2 / 2}$$

we have

$$\begin{aligned}
\text{Cov}(\Delta_R(\Gamma_i), \Delta_R(\Gamma_j)) &= \int_{\Gamma_i} \int_{\Gamma_j} \frac{\partial^2}{\partial \hat{n}_i \partial \hat{n}_j} \left(\text{Cov} \left(\log |\hat{f}_R(z_j)|, \log |\hat{f}_R(z_i)| \right) + \frac{\gamma^2}{4} \right) |dz_j| |dz_i| \\
&= \frac{1}{4} \int_{\Gamma_i} \int_{\Gamma_j} \frac{\partial^2}{\partial \hat{n}_i \partial \hat{n}_j} \text{Li}_2(|\hat{K}_R(z_j, z_i)|^2) |dz_j| |dz_i| \\
&= \frac{1}{4} \int_{\Gamma_i} \int_{\Gamma_j} \frac{\partial^2}{\partial \hat{n}_i \partial \hat{n}_j} \sum_{\alpha=1}^{\infty} \frac{e^{-R^2 \alpha |z_j - z_i|^2}}{\alpha^2} |dz_j| |dz_i|.
\end{aligned}$$

(Note that since Li_2 is not differentiable at 1, the integrand is still only defined for $z_j \neq z_i$.)

Lemma 13.

$$\frac{1}{R} \int_{\Gamma_i} \int_{\Gamma_j} \left| \frac{\partial^2}{\partial \hat{n}_i \partial \hat{n}_j} e^{-R^2 |z_j - z_i|^2} \right| |dz_j| |dz_i| = O(1)$$

where the implicit constant depends only on Γ_i and Γ_j .

Lemma 14.

$$\frac{1}{R} \int_{\Gamma_i} \int_{\Gamma_j} \frac{\partial^2}{\partial \hat{n}_i \partial \hat{n}_j} e^{-R^2 |z_j - z_i|^2} |dz_j| |dz_i| \rightarrow 2\sqrt{\pi} \mathcal{L}(\Gamma_i, \Gamma_j)$$

as $R \rightarrow \infty$.

We postpone the proofs of these lemmas, and proceed. Since the power series defining Li_2 is absolutely convergent on the unit disc we may differentiate termwise to obtain

$$\frac{\partial^2}{\partial \hat{n}_i \partial \hat{n}_j} \sum_{\alpha=1}^{\infty} \frac{e^{-R^2 \alpha |z_j - z_i|^2}}{\alpha^2} = \sum_{\alpha=1}^{\infty} \frac{1}{\alpha^2} \frac{\partial^2}{\partial \hat{n}_i \partial \hat{n}_j} e^{-R^2 \alpha |z_j - z_i|^2}$$

for all $z_j \neq z_i$. This implies that, using Lemma 13 and dominated convergence,

$$\begin{aligned}
\lim_{R \rightarrow \infty} \frac{1}{R} \text{Cov}(\Delta_R(\Gamma_i), \Delta_R(\Gamma_j)) &= \frac{1}{4} \sum_{\alpha=1}^{\infty} \left(\frac{1}{\alpha^2} \lim_{R \rightarrow \infty} \frac{1}{R} \int_{\Gamma_i} \int_{\Gamma_j} \frac{\partial^2}{\partial \hat{n}_i \partial \hat{n}_j} e^{-R^2 \alpha |z_j - z_i|^2} |dz_j| |dz_i| \right) \\
&\stackrel{\text{Lemma 14}}{=} \frac{1}{4} \sum_{\alpha=1}^{\infty} \left(\frac{1}{\alpha^2} \sqrt{\alpha} \cdot 2\sqrt{\pi} \mathcal{L}(\Gamma_i, \Gamma_j) \right) \\
&= \frac{\sqrt{\pi}}{2} \left(\sum_{\alpha=1}^{\infty} \frac{1}{\alpha^{3/2}} \right) \mathcal{L}(\Gamma_i, \Gamma_j)
\end{aligned}$$

which is (13). It remains to prove Lemmas 13 and 14.

2.1.1. Proof of Lemmas 13 and 14. First note that

$$\frac{\partial^2}{\partial \hat{n}_i \partial \hat{n}_j} e^{-R^2 |z_j - z_i|^2} = e^{-R^2 |z_j - z_i|^2} (2R^2 \langle \hat{n}_i, \hat{n}_j \rangle - 4R^4 \langle \hat{n}_i, z_j - z_i \rangle \langle \hat{n}_j, z_j - z_i \rangle).$$

We will show that

$$(17) \quad R \int_{\Gamma_i} \int_{\Gamma_j} e^{-R^2|z_j - z_i|^2} |dz_j| |dz_i| = O(1),$$

$$(18) \quad R \int_{\Gamma_i} \int_{\Gamma_j} \langle \hat{n}_i, \hat{n}_j \rangle e^{-R^2|z_j - z_i|^2} |dz_j| |dz_i| \rightarrow \sqrt{\pi} \mathcal{L}(\Gamma_i, \Gamma_j),$$

$$(19) \quad R^3 \int_{\Gamma_i} \int_{\Gamma_j} |z_j - z_i|^2 e^{-R^2|z_j - z_i|^2} |dz_j| |dz_i| = O(1)$$

and that

$$(20) \quad R^3 \int_{\Gamma_i} \int_{\Gamma_j} \langle \hat{n}_i, z_j - z_i \rangle \langle \hat{n}_j, z_j - z_i \rangle e^{-R^2|z_j - z_i|^2} |dz_j| |dz_i| \rightarrow 0.$$

This will yield both lemmas, and therefore (4).

With this in mind, we define, for $z_i \in \Gamma_i$,

$$I_R(z_i) = R \int_{\Gamma_j} \langle \hat{n}_i, \hat{n}_j \rangle e^{-R^2|z_j - z_i|^2} |dz_j|,$$

$$I'_R(z_i) = R \int_{\Gamma_j} e^{-R^2|z_j - z_i|^2} |dz_j|,$$

$$J_R(z_i) = R^3 \int_{\Gamma_j} \langle \hat{n}_i, z_j - z_i \rangle \langle \hat{n}_j, z_j - z_i \rangle e^{-R^2|z_j - z_i|^2} |dz_j|$$

and

$$J'_R(z_i) = R^3 \int_{\Gamma_j} |z_j - z_i|^2 e^{-R^2|z_j - z_i|^2} |dz_j|.$$

Fix $\beta > 3$, define

$$\varepsilon_R = \frac{\sqrt{\beta \log R}}{R}$$

and split

$$\Gamma_i = \Gamma'_i \cup \Gamma''_i \cup (\Gamma_i \cap \Gamma_j)$$

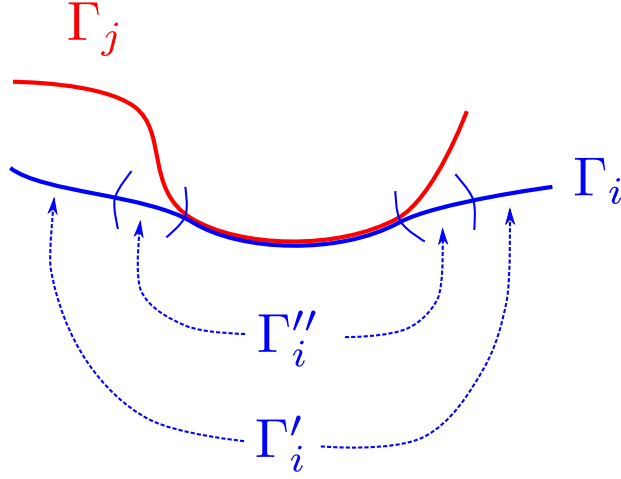
where

$$\Gamma'_i = \{z_i \in \Gamma_i : d(z_i, \Gamma_j) \geq \varepsilon_R\}$$

and

$$\Gamma''_i = \{z_i \in \Gamma_i : 0 < d(z_i, \Gamma_j) < \varepsilon_R\}$$

(see Figure 2); these sets (and all of the sets we define subsequently) may be empty.

FIGURE 2. Illustration of Γ'_i and Γ''_i **Estimating J'_R .**

We begin by estimating J'_R . We estimate separately the integral of J'_R over each of the sets Γ'_i , Γ''_i and $\Gamma_i \cap \Gamma_j$. Trivially

$$\int_{\Gamma'_i} J'_R(z_i) |dz_i| \leq CR^{3-\beta}$$

where the constant C depends only on Γ_i and Γ_j . Next, for $z_i \in \Gamma''_i$, denote by z_i^* the closest point on Γ_j to z_i (if there is more than one such point, we choose one arbitrarily). Fix $z_i \in \Gamma''_i$ and define the points in Γ_j that are “far” from z_i^* by

$$\Gamma_j^{(F)}(z_i) = \left\{ z_j \in \Gamma_j : |z_j - z_i^*| > \frac{2}{R} \right\}$$

and the “nearby” points by

$$\Gamma_j^{(N)}(z_i) = \left\{ z_j \in \Gamma_j : |z_j - z_i^*| \leq \frac{2}{R} \right\};$$

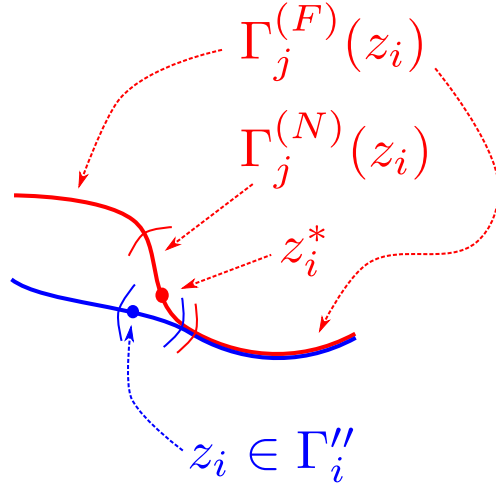
see Figure 3. We split

$$J'_R(z_i) = R \left(\int_{\Gamma_j^{(F)}} + \int_{\Gamma_j^{(N)}} \right) R^2 |z_j - z_i|^2 e^{-R^2 |z_j - z_i|^2} |dz_j|$$

and estimate each integral separately. Note that

$$R \int_{\Gamma_j^{(N)}} R^2 |z_j - z_i|^2 e^{-R^2 |z_j - z_i|^2} |dz_j| \lesssim R \text{length}(\Gamma_j^{(N)}) = O(1).$$

Note also that $|z_j - z_i| \geq |z_j - z_i^*| - |z_i^* - z_i| \geq |z_j - z_i^*| - |z_j - z_i|$, for any $z_j \in \Gamma_j$, by the definition of z_i^* . This implies that $R|z_j - z_i| \geq \frac{R}{2}|z_j - z_i^*| > 1$ for $z_i \in \Gamma''_i$ and $z_j \in \Gamma_j^{(F)}$ and so,

FIGURE 3. Illustration of $\Gamma_j^{(F)}(z_i)$ and $\Gamma_j^{(N)}(z_i)$

since the function $t \mapsto t^2 e^{-t^2}$ is decreasing for $t > 1$, we have

$$R \int_{\Gamma_j^{(F)}} R^2 |z_j - z_i|^2 e^{-R^2 |z_j - z_i|^2} |dz_j| \leq \frac{R}{4} \int_{\Gamma_j^{(F)}} R^2 |z_j - z_i^*|^2 e^{-\frac{R^2}{4} |z_j - z_i^*|^2} |dz_j|.$$

We now use some ‘‘Laplace type estimates’’ to bound the integral on the right-hand side of this previous inequality. Let $\gamma_j: [0, 1] \rightarrow \mathbb{C}$ be a parameterisation of the curve Γ_j satisfying $0 < m \leq |\dot{\gamma}(t)| \leq M$ for all $t \in [0, 1]$. Since Γ_j is simple, we see that there exists $m' > 0$ such that $m'|t - s| \leq |\gamma_j(t) - \gamma_j(s)| \leq M|t - s|$. Denote by t^* the (unique) value such that $\gamma_j(t^*) = z_i^*$ and note that⁷

$$\Gamma_j^{(F)}(z_i) \subseteq \left\{ z_j = \gamma_j(t) : |t - t^*| > \frac{2}{MR} \right\}.$$

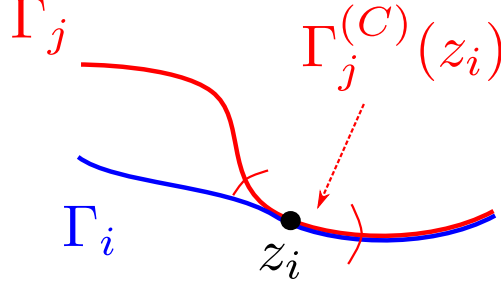
We thus have

$$\begin{aligned} R \int_{\Gamma_j^{(F)}} R^2 |z_j - z_i^*|^2 e^{-\frac{R^2}{4} |z_j - z_i^*|^2} |dz_j| &\leq R \int_{|t - t^*| > \frac{2}{MR}} R^2 |\gamma_j(t) - \gamma_j(t^*)|^2 e^{-\frac{R^2}{4} |\gamma_j(t) - \gamma_j(t^*)|^2} |\dot{\gamma}_j(t)| dt \\ &\leq R \int_{|t - t^*| > \frac{2}{MR}} R^2 M^2 (t - t^*)^2 e^{-\frac{R^2}{4} (m')^2 (t - t^*)^2} M dt, \end{aligned}$$

and making the change of variables $s = \frac{R}{2} m' (t - t^*)$ we have

$$R \int_{\Gamma_j^{(F)}} R^2 |z_j - z_i^*|^2 e^{-4R^2 |z_j - z_i^*|^2} |dz_j| \leq \frac{8M^3}{(m')^3} \int_{|s| > \frac{m'}{M}} s^2 e^{-s^2} ds = O(1).$$

⁷Strictly speaking we should write $\{z_j = \gamma_j(t) : 0 \leq t \leq 1 \text{ and } |t - t^*| > \frac{2}{MR}\}$; we shall frequently ignore this issue, as it will not affect our upper bounds.

FIGURE 4. Illustration of $\Gamma_j^{(C)}(z_i)$

This implies that

$$\int_{\Gamma_i''} J'_R(z_i) |dz_i| \lesssim \text{length}(\Gamma_i'') \rightarrow 0$$

as $R \rightarrow \infty$, since the set Γ_i'' decreases to the empty set as $\varepsilon_R \rightarrow 0$. We thus have

$$(21) \quad \int_{\Gamma_i' \cup \Gamma_i''} J'_R(z_i) |dz_i| \rightarrow 0.$$

We now bound $\int_{\Gamma_i \cap \Gamma_j} J'_R(z_i) |dz_i|$. Fixing z_i , it is clear that we may ignore the points $z_j \in \Gamma_j$ where $|z_j - z_i| \geq \varepsilon_R$, since their contribution is uniformly negligible. Denote the points “close” to z_i by $\Gamma_j^{(C)}(z_i) = \{z_j \in \Gamma_j : |z_j - z_i| \leq \varepsilon_R\}$, see Figure 4.

Let γ_j be the same parameterisation of Γ_j as before and let $\tau(z_i)$ be the (unique) value such that $\gamma_j(\tau(z_i)) = z_i$. Arguing similarly we get

$$(22) \quad \begin{aligned} & R \int_{\Gamma_j^{(C)}} R^2 |z_j - z_i|^2 e^{-R^2 |z_j - z_i|^2} |dz_j| \\ & \leq R \int_{|t - \tau(z_i)| \leq \frac{\varepsilon_R}{m'}} R^2 |\gamma_j(t) - \gamma_j(\tau(z_i))|^2 e^{-R^2 |\gamma_j(t) - \gamma_j(\tau(z_i))|^2} |\dot{\gamma}_j(t)| dt \\ & \leq \left(\frac{M}{m'}\right)^3 \int_{|s| \leq R\varepsilon_R} s^2 e^{-s^2} ds = O(1), \end{aligned}$$

which shows that $J'_R(z_i)$ is bounded for $z_i \in \Gamma_i \cap \Gamma_j$ and so, when combined with (21) proves (19).

Estimating J_R .

We next show (20). Since trivially $|J_R(z_i)| \leq J'_R(z_i)$ we see that (21) implies that

$$\left| \int_{\Gamma_i' \cup \Gamma_i''} J_R(z_i) |dz_i| \right| \leq \int_{\Gamma_i' \cup \Gamma_i''} J'_R(z_i) |dz_i| \rightarrow 0 \quad \text{as } \varepsilon_R \rightarrow 0.$$

Furthermore, for a fixed $z_i \in \Gamma_i \cap \Gamma_j$, as we noted previously

$$R \int_{\Gamma_j \setminus \Gamma_j^{(C)}} R^2 |z_j - z_i|^2 e^{-R^2 |z_j - z_i|^2} |dz_j|$$

is uniformly negligible. Finally note that for $z_i \in \Gamma_i \cap \Gamma_j$ and $z_j \in \Gamma_j^{(C)}(z_i)$,

$$\begin{aligned} \langle \hat{n}_j(z_j), z_j - z_i \rangle &= |z_j - z_i| \left\langle \hat{n}_j(z_j), \frac{z_j - z_i}{|z_j - z_i|} \right\rangle \\ &= |z_j - z_i| \langle \hat{n}_j(z_i), \hat{\tau}_j(z_i) \rangle (1 + o(1)) = o(|z_j - z_i|) \quad \text{as } \varepsilon_R \rightarrow 0 \end{aligned}$$

where $\hat{\tau}_j$ denotes the unit tangent vector to Γ_j , and the estimate is uniform in z_i . This implies that for $z_i \in \Gamma_i \cap \Gamma_j$ we have $|J_R(z_i)| = o(J'_R(z_i)) = o(1)$, by (22). This proves (20).

Estimating I'_R .

We next show (17), the argument is similar to the proof of (19). It is again easy to see that

$$\int_{\Gamma'_i} I'_R(z_i) |dz_i| \leq C R^{1-\beta} \rightarrow 0$$

while, for $z_i \in \Gamma''_i$, using the same notation as before, since $|z_j - z_i| \geq \frac{1}{2}|z_j - z_i^*|$ we get

$$\begin{aligned} I'_R(z_i) &= R \int_{\Gamma_j} e^{-R^2 |z_j - z_i|^2} |dz_j| \leq R \int_{\Gamma_j} e^{-\frac{R^2}{4} |z_j - z_i^*|^2} |dz_j| \\ &= R \int_0^1 e^{-\frac{R^2}{4} |\gamma_j(t) - \gamma_j(t^*)|^2} |\dot{\gamma}_j(t)| dt \leq \frac{2M}{m'} \int_{-\infty}^{\infty} e^{-s^2} ds = O(1). \end{aligned}$$

We therefore have

$$\int_{\Gamma''_i} I'_R(z_i) |dz_i| \leq C \text{length}(\Gamma''_i) \rightarrow 0 \quad \text{as } \varepsilon_R \rightarrow 0.$$

Finally, for $z_i \in \Gamma_i \cap \Gamma_j$, using again the same notation, it is easy to see once more that the contribution to $I'_R(z_i)$ of $\Gamma_j \setminus \Gamma_j^{(C)}(z_i)$ is negligible and that

$$\begin{aligned} R \int_{\Gamma_j^{(C)}} e^{-R^2 |z_j - z_i|^2} |dz_j| &\leq R \int_{|t - \tau(z_i)| \leq \frac{\varepsilon_R}{m'}} e^{-R^2 |\gamma_j(t) - \gamma_j(\tau(z_i))|^2} |\dot{\gamma}_j(t)| dt \\ &\leq \frac{M}{m'} \int_{|s| \leq R\varepsilon_R} e^{-s^2} ds = O(1). \end{aligned}$$

We have shown that $I'_R(z_i) = O(1)$ for $z_i \in \Gamma_i \cap \Gamma_j$, which proves (17).

Asymptotic for I_R .

It remains to prove (18). Note that

$$\left| \int_{\Gamma'_i \cup \Gamma''_i} I_R(z_i) |dz_i| \right| \leq \int_{\Gamma'_i \cup \Gamma''_i} I'_R(z_i) |dz_i| \rightarrow 0 \quad \text{as } \varepsilon_R \rightarrow 0$$

and so it remains only to compute

$$\int_{\Gamma_i \cap \Gamma_j} I_R(z_i) |dz_i|.$$

If the curve Γ_j is not closed we define z_j^+ and z_j^- to be the endpoints of the curve and $\Gamma_j^\pm = \{z_j \in \Gamma_j : |z_j - z_j^\pm| < \varepsilon_R\}$; if the curve is closed we define these sets to be empty. Note once more that

$$\left| \int_{\Gamma_i \cap \Gamma_j \cap \Gamma_j^\pm} I_R(z_i) |dz_i| \right| \leq \int_{\Gamma_i \cap \Gamma_j \cap \Gamma_j^\pm} I'_R(z_i) |dz_i| \lesssim \text{length}(\Gamma_j^\pm) \rightarrow 0 \quad \text{as } \varepsilon_R \rightarrow 0.$$

For $z_i \in (\Gamma_i \cap \Gamma_j) \setminus (\Gamma_j^+ \cup \Gamma_j^-)$ recall that $\Gamma_j^{(C)}(z_i) = \{z_j \in \Gamma_j : |z_j - z_i| \leq \varepsilon_R\}$ and note that for $z_j \in \Gamma_j^{(C)}(z_i)$ we have $|\hat{n}_j(z_j) - \hat{n}_j(z_i)| = o(1)$, where the term $o(1)$ is uniform in z_i . We therefore have

$$\begin{aligned} I_R(z_i) &= R \int_{\Gamma_j^{(C)}(z_i)} \langle \hat{n}_i(z_i), \hat{n}_j(z_j) \rangle e^{-R^2|z_j - z_i|^2} |dz_j| \\ &\quad + R \int_{\Gamma_j \setminus \Gamma_j^{(C)}(z_i)} \langle \hat{n}_i(z_i), \hat{n}_j(z_i) \rangle e^{-R^2|z_j - z_i|^2} |dz_j| \\ (23) \quad &= R \langle \hat{n}_i(z_i), \hat{n}_j(z_i) \rangle \int_{\Gamma_j^{(C)}(z_i)} e^{-R^2|z_j - z_i|^2} |dz_j| (1 + o(1)) + O(R^{1-\beta}), \end{aligned}$$

and we shall compute the asymptotics of this last integral using (more accurate) ‘‘Laplace type estimates’’.

Let $\gamma_j : [0, 1] \rightarrow \mathbb{C}$ be the same parameterisation of Γ_j as before, and let $\tau(z_i)$ be the value such that $\gamma_j(\tau(z_i)) = z_i$. Note that

$$\left\{ z_j = \gamma_j(t) : |t - \tau(z_i)| \leq \frac{\varepsilon_R}{M} \right\} \subseteq \Gamma_j^{(C)}(z_i) \subseteq \left\{ z_j = \gamma_j(t) : |t - \tau(z_i)| \leq \frac{\varepsilon_R}{m'} \text{ and } 0 < t < 1 \right\}.$$

We remark here that, if $\{t_+, t_-\} = \{0, 1\}$, then $\{\gamma_j(t_+), \gamma_j(t_-)\} = \{z_j^+, z_j^-\}$. Since

$$\varepsilon_R \leq |\gamma_j(\tau(z_i)) - \gamma_j(t_\pm)| \leq M|\tau(z_i) - t_\pm|$$

we have $\tau(z_i) \geq \varepsilon_R/M$ and $\tau(z_i) \leq 1 - \varepsilon_R/M$ and so the range $|t - \tau(z_i)| \leq \frac{\varepsilon_R}{M}$ is contained in $[0, 1]$. The above implies that, given $0 < \delta < 1$, for large enough R (uniformly in $\tau(z_i)$) we have

$$(1 - \delta)|\dot{\gamma}_j(\tau(z_i))||t - \tau(z_i)| \leq |\gamma_j(t) - \gamma_j(\tau(z_i))| \leq (1 + \delta)|\dot{\gamma}_j(\tau(z_i))||t - \tau(z_i)|$$

and

$$(1 - \delta)|\dot{\gamma}_j(\tau(z_i))| \leq |\dot{\gamma}_j(t)| \leq (1 + \delta)|\dot{\gamma}_j(\tau(z_i))|$$

for t such that $\gamma_j(t) \in \Gamma_j^{(C)}(z_i)$. This implies that, defining

$$\tau_+ = \min\{1, \tau(z_i) + \frac{\varepsilon_R}{m'}\} \quad \text{and} \quad \tau_- = \max\{0, \tau(z_i) - \frac{\varepsilon_R}{m'}\},$$

we have

$$\begin{aligned} \int_{\Gamma_j^{(C)}(z_i)} e^{-R^2|z_j-z_i|^2} |dz_j| &\leq \int_{\tau_-}^{\tau_+} e^{-R^2|\gamma_j(t)-\gamma_j(\tau(z_i))|^2} |\dot{\gamma}_j(t)| dt \\ &\leq (1+\delta) \int_{\tau_-}^{\tau_+} e^{-R^2(1-\delta)^2|\dot{\gamma}_j(\tau(z_i))|^2(t-\tau(z_i))^2} |\dot{\gamma}_j(\tau(z_i))| dt \end{aligned}$$

and the change of variables $s = R(1-\delta)|\dot{\gamma}_j(\tau(z_i))|(t-\tau(z_i))$ yields

$$\int_{\Gamma_j^{(C)}(z_i)} e^{-R^2|z_j-z_i|^2} |dz_j| \leq (1+\delta) \int_{R(1-\delta)|\dot{\gamma}_j(\tau(z_i))|(\tau_--\tau(z_i))}^{R(1-\delta)|\dot{\gamma}_j(\tau(z_i))|(\tau_+-\tau(z_i))} e^{-s^2} \frac{ds}{R(1-\delta)}.$$

Notice that

$$\tau_+ - \tau(z_i) = \min\{1 - \tau(z_i), \frac{\varepsilon_R}{m'}\} \geq \frac{\varepsilon_R}{M} \quad \text{and} \quad \tau_- - \tau(z_i) = \max\{-\tau(z_i), -\frac{\varepsilon_R}{m'}\} \leq -\frac{\varepsilon_R}{M}.$$

Since $R\varepsilon_R \rightarrow \infty$ (and $|\dot{\gamma}_j(\tau(z_i))| \neq 0$) the right hand side of the previous displayed expression equals

$$\frac{(1+\delta)}{R(1-\delta)} \int_{\mathbb{R}} e^{-s^2} ds (1+o(1)) = \frac{(1+\delta)\sqrt{\pi}}{R(1-\delta)} (1+o(1)).$$

Similar computations yield

$$\int_{\Gamma_j^{(C)}(z_i)} e^{-R^2|z_j-z_i|^2} |dz_j| \geq \frac{(1-\delta)\sqrt{\pi}}{R(1+\delta)} (1+o(1))$$

and since δ is arbitrary we conclude that

$$\int_{\Gamma_j^{(C)}(z_i)} e^{-R^2|z_j-z_i|^2} |dz_j| = \frac{\sqrt{\pi}}{R} (1+o(1)).$$

Combining this with (23), and discarding the integration over $\Gamma_j \setminus \Gamma_j^{(C)}(z_i)$, we have

$$I_R(z_i) = \sqrt{\pi} \langle \hat{n}_i(z_i), \hat{n}_j(z_i) \rangle (1+o(1)),$$

where the term $o(1)$ is uniform in z_i . We conclude that

$$\begin{aligned} \int_{\Gamma_i} I_R(z_i) |dz_i| &= \sqrt{\pi} \int_{(\Gamma_i \cap \Gamma_j) \setminus (\Gamma_j^+ \cup \Gamma_j^-)} \langle \hat{n}_i(z_i), \hat{n}_j(z_i) \rangle |dz_i| (1+o(1)) \\ &= \sqrt{\pi} \left(\mathcal{L}(\Gamma_i, \Gamma_j) - \int_{(\Gamma_i \cap \Gamma_j) \cap (\Gamma_j^+ \cup \Gamma_j^-)} \langle \hat{n}_i(z_i), \hat{n}_j(z_i) \rangle |dz_i| \right) (1+o(1)) \\ &= (\sqrt{\pi} + o(1)) \mathcal{L}(\Gamma_i, \Gamma_j), \end{aligned}$$

which is (18). This completes the proof of the lemmas, and therefore of (4).

3. ASYMPTOTIC NORMALITY

In this section we show that $\Delta_R(\Gamma)$ is asymptotically normal, which will complete the proof of Theorem 1. We first define a random variable $\Delta_R^{(m)}$ that approximates $\Delta_R(\Gamma)$ in $L^2(\mathbb{P})$ and then prove a CLT for $\Delta_R^{(m)}$. Specifically, defining $\bar{\Delta}_R = \Delta_R(\Gamma) - \mathbb{E}[\Delta_R(\Gamma)]$, we will show that:

- There exists R_0 such that

$$(24) \quad \mathbb{E}[(\bar{\Delta}_R - \Delta_R^{(m)})^2] \leq C \frac{R}{\sqrt{m}}$$

for all $R \geq R_0$ and $m \geq 1$.

- For each fixed m

$$(25) \quad \frac{\Delta_R^{(m)}}{\sqrt{\text{Var } \Delta_R^{(m)}}} \rightarrow \mathcal{N}_{\mathbb{R}}(0, 1)$$

in distribution, as $R \rightarrow \infty$.

When combined with our previous asymptotic for the variance, this allows us to conclude asymptotic normality for $\Delta_R(\Gamma)$, by a standard argument. We begin by defining the approximant $\Delta_R^{(m)}$.

3.1. Definition of approximant. Recall that

$$\bar{\Delta}_R = \sum_{i=1}^N a_i \int_{\Gamma_i} \frac{\partial}{\partial \hat{n}_i} \log |\hat{f}_R(z_i)| |dz_i|,$$

this follows from (14) and (15). We will define

$$\Delta_R^{(m)} = \sum_{i=1}^N a_i \Delta_i^{(m)} = \sum_{i=1}^N a_i \int_{\Gamma_i} \frac{\partial}{\partial \hat{n}_i} \log_m |\hat{f}_R(z_i)| |dz_i|$$

where \log_m is a polynomial that approximates \log in an appropriate sense. To this end we recall the Wiener chaos decomposition (sometimes called the Hermite-Itô expansion) of $L^2(\mu)$ where $d\mu(z) = \frac{1}{\pi} e^{-|z|^2} dm(z)$ is the Gaussian measure on the plane; for a more comprehensive treatment we refer the reader to [Jan97, Chapters 2 and 3].

Let \mathcal{P}_m denote the subspace of $L^2(\mu)$ given by polynomials (in the variables z and \bar{z}) of degree at most m , and denote $H^{:0:} = \mathcal{P}_0$ and $H^{:m:} = \mathcal{P}_m \ominus \mathcal{P}_{m-1}$ for $m \geq 1$. Given a monomial $\zeta^\alpha \bar{\zeta}^\beta$ with $\alpha + \beta = m$ we write $:\zeta^\alpha \bar{\zeta}^\beta:$ to denote its projection to $H^{:m:}$, which is usually called a Wick product. A computation (see [Jan97, Example 3.32]) shows that the set of all Wick products $:\zeta^\alpha \bar{\zeta}^\beta:$ with $\alpha + \beta = m$ is an orthogonal basis for $H^{:m:}$, and moreover $\|:\zeta^\alpha \bar{\zeta}^\beta:\|^2 = \alpha! \beta!$ (the norm here is the norm inherited from $L^2(\mu)$). Furthermore [Jan97, Theorem 2.6]

$$L^2(\mu) = \bigoplus_{m=0}^{\infty} H^{:m:}.$$

We now expand $\log |\zeta|$ in terms of this orthonormal basis. Since the function is radial, only the terms with $\alpha = \beta$ contribute, and a calculation [NS11, Lemma 2.1] yields

$$(26) \quad \log |\zeta| = -\frac{\gamma}{2} + \sum_{\alpha=1}^{\infty} \frac{c_{\alpha}}{\alpha!} :|\zeta|^{2\alpha}:$$

where $c_{\alpha} = \frac{(-1)^{\alpha+1}}{2\alpha}$.

Remark. We may alternatively interpret (26) as an expansion of the logarithm in terms of Laguerre polynomials, by noting that $:|\zeta|^{2\alpha} := (-1)^{\alpha} L_{\alpha}(|\zeta|^2)$ where $L_{\alpha}(x) = e^x \frac{d^{\alpha}}{dx^{\alpha}} (x^{\alpha} e^{-x})$.

We finally define

$$\log_m |\zeta| = -\frac{\gamma}{2} + \sum_{\alpha=1}^m \frac{c_{\alpha}}{\alpha!} :|\zeta|^{2\alpha}:$$

which defines $\Delta_R^{(m)}$. Note that $\log_m |\widehat{f}_R(z)|$ approximates $\log |\widehat{f}_R(z)|$ in $L^2(\mathbb{P})$.

3.2. Quantifying the approximation. We first define

$$\Delta_i = \int_{\Gamma_i} \frac{\partial}{\partial \hat{n}_i} \log |\widehat{f}_R(z_i)| |dz_i|$$

and note that

$$\mathbb{E}[(\overline{\Delta}_R - \Delta_R^{(m)})^2] = \sum_{i,j=1}^N a_i a_j \mathbb{E}[(\Delta_i - \Delta_i^{(m)})(\Delta_j - \Delta_j^{(m)})].$$

We have already computed (see (16)) that

$$\mathbb{E}[\Delta_i \Delta_j] = \int_{\Gamma_i} \int_{\Gamma_j} \frac{\partial^2}{\partial \hat{n}_i \partial \hat{n}_j} \mathbb{E} \left[\log |\widehat{f}_R(z_j)| \log |\widehat{f}_R(z_i)| \right] |dz_j| |dz_i|.$$

Also, since

$$\mathbb{E}[\Delta_i \Delta_j^{(m)}] = \mathbb{E} \left[\int_{\Gamma_i} \int_{\Gamma_j} \frac{\partial^2}{\partial \hat{n}_i \partial \hat{n}_j} \log_m |\widehat{f}_R(z_j)| \log |\widehat{f}_R(z_i)| |dz_j| |dz_i| \right],$$

Lemma 10 allows us to apply Lemma 8 to see that

$$\mathbb{E}[\Delta_i \Delta_j^{(m)}] = \int_{\Gamma_i} \int_{\Gamma_j} \frac{\partial^2}{\partial \hat{n}_i \partial \hat{n}_j} \mathbb{E} \left[\log_m |\widehat{f}_R(z_j)| \log |\widehat{f}_R(z_i)| \right] |dz_j| |dz_i|.$$

Arguing identically, but using Lemma 9 (with $N = 2$) instead of Lemma 10, we get

$$(27) \quad \mathbb{E}[\Delta_i^{(m)} \Delta_j^{(m)}] = \int_{\Gamma_i} \int_{\Gamma_j} \frac{\partial^2}{\partial \hat{n}_i \partial \hat{n}_j} \mathbb{E} \left[\log_m |\widehat{f}_R(z_j)| \log_m |\widehat{f}_R(z_i)| \right] |dz_j| |dz_i|.$$

We conclude that

$$\begin{aligned}
\mathbb{E}[(\Delta_i - \Delta_i^{(m)})(\Delta_j - \Delta_j^{(m)})] &= \int_{\Gamma_i} \int_{\Gamma_j} \frac{\partial^2}{\partial \hat{n}_i \partial \hat{n}_j} \mathbb{E}[(\log |\hat{f}_R(z_j)| - \log_m |\hat{f}_R(z_j)|) \cdot \\
&\quad \cdot (\log |\hat{f}_R(z_i)| - \log_m |\hat{f}_R(z_i)|)] |dz_j| |dz_i| \\
&= \int_{\Gamma_i} \int_{\Gamma_j} \frac{\partial^2}{\partial \hat{n}_i \partial \hat{n}_j} \mathbb{E}\left[\left(\sum_{\alpha_j > m} \frac{c_{\alpha_j}}{\alpha_j!} :|\hat{f}_R(z_j)|^{2\alpha_j}: \right) \cdot \left(\sum_{\alpha_i > m} \frac{c_{\alpha_i}}{\alpha_i!} :|\hat{f}_R(z_i)|^{2\alpha_i}: \right)\right] |dz_j| |dz_i|.
\end{aligned}$$

Now since the expansions inside the expectation are valid in $L^2(\mathbb{P})$ we have

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{\alpha_j > m} \frac{c_{\alpha_j}}{\alpha_j!} :|\hat{f}_R(z_j)|^{2\alpha_j}: \right) \left(\sum_{\alpha_i > m} \frac{c_{\alpha_i}}{\alpha_i!} :|\hat{f}_R(z_i)|^{2\alpha_i}: \right)\right] \\
= \sum_{\alpha_i, \alpha_j > m} \frac{c_{\alpha_i} c_{\alpha_j}}{\alpha_i! \alpha_j!} \mathbb{E}\left[:|\hat{f}_R(z_j)|^{2\alpha_j}: :|\hat{f}_R(z_i)|^{2\alpha_i}: \right].
\end{aligned}$$

Now, by [Jan97, Theorem 3.9], we have

$$(28) \quad \mathbb{E}\left[:|\hat{f}_R(z_j)|^{2\alpha_j}: :|\hat{f}_R(z_i)|^{2\alpha_i}: \right] = \begin{cases} \alpha_i! \alpha_j! |\hat{K}_R(z_j, z_i)|^{2\alpha_i} & \text{if } \alpha_i = \alpha_j \\ 0 & \text{otherwise} \end{cases}$$

which yields

$$\begin{aligned}
\mathbb{E}[(\Delta_i - \Delta_i^{(m)})(\Delta_j - \Delta_j^{(m)})] &= \int_{\Gamma_i} \int_{\Gamma_j} \frac{\partial^2}{\partial \hat{n}_i \partial \hat{n}_j} \sum_{\alpha > m} c_\alpha^2 |\hat{K}_R(z_j, z_i)|^{2\alpha} |dz_j| |dz_i| \\
&= \frac{1}{4} \int_{\Gamma_i} \int_{\Gamma_j} \frac{\partial^2}{\partial \hat{n}_i \partial \hat{n}_j} \sum_{\alpha > m} \frac{1}{\alpha^2} e^{-2\alpha R^2 |z_j - z_i|^2} |dz_j| |dz_i|.
\end{aligned}$$

Remark. The identity (28) together with (26) essentially proves Lemma 12.

Using Lemma 13, we have

$$\begin{aligned}
|\mathbb{E}[(\Delta_i - \Delta_i^{(m)})(\Delta_j - \Delta_j^{(m)})]| &\leq \frac{1}{4} \int_{\Gamma_i} \int_{\Gamma_j} \left| \frac{\partial^2}{\partial \hat{n}_i \partial \hat{n}_j} \sum_{\alpha > m} \frac{1}{\alpha^2} e^{-2\alpha R^2 |z_j - z_i|^2} \right| |dz_j| |dz_i| \\
&\leq CR \sum_{\alpha > m} \frac{1}{\alpha^{3/2}} \leq C \frac{R}{\sqrt{m}}.
\end{aligned}$$

Finally

$$\mathbb{E}[(\bar{\Delta}_R - \Delta_R^{(m)})^2] \leq \sum_{i,j=1}^N a_i a_j \left| \mathbb{E}[(\Delta_i - \Delta_i^{(m)})(\Delta_j - \Delta_j^{(m)})] \right| \leq C \frac{R}{\sqrt{m}},$$

which is (24).

3.3. CLT for the approximant. We finish by proving (25). We claim that it's enough to prove that for any non-negative integers p_1, \dots, p_N we have, as $R \rightarrow \infty$,

$$(29) \quad \mathbb{E} \left[\prod_{i=1}^N (\Delta_i^{(m)})^{p_i} \right] = \mathbb{E} \left[\prod_{i=1}^N \xi_i^{p_i} \right] + o(R^{P/2})$$

where ξ_i is a sequence of jointly (real) Gaussian random variables, with mean 0 and covariance

$$\text{Cov}(\xi_i, \xi_j) = \mathbb{E}[\xi_i \xi_j] = \text{Cov}(\Delta_i^{(m)}, \Delta_j^{(m)}),$$

and $P = p_1 + \dots + p_N$.

Remark. Notice that

$$\text{Var } \Delta_i^{(m)} = \frac{\sqrt{\pi}}{2} \left(\sum_{\alpha=1}^m \frac{1}{\alpha^{3/2}} \right) \mathcal{L}(\Gamma_i, \Gamma_i) R(1 + o(1)) \quad \text{as } R \rightarrow \infty,$$

which follows from combining (27) and Lemma 14. By hypothesis, Γ is a non-zero \mathbb{R} -chain and so we may assume that $\mathcal{L}(\Gamma_i, \Gamma_i) \neq 0$ for each i , which means that

$$\prod_{i=1}^N (\text{Var } \Delta_i^{(m)})^{p_i/2} \simeq R^{P/2}.$$

Now, defining $\widehat{\xi}_i = \frac{\xi_i}{\sqrt{\text{Var } \xi_i}} = \frac{\xi_i}{\sqrt{\text{Var } \Delta_i^{(m)}}}$, we see that (29) is equivalent to

$$\mathbb{E} \left[\prod_{i=1}^N \left(\frac{\Delta_i^{(m)}}{\sqrt{\text{Var } \Delta_i^{(m)}}} \right)^{p_i} \right] = \mathbb{E} \left[\prod_{i=1}^N \widehat{\xi}_i^{p_i} \right] + o(1),$$

and note that $\left| \mathbb{E} \left[\prod_{j=1}^N \widehat{\xi}_j^{p_j} \right] \right|$ is a bounded quantity.

To see that it suffices to show (29), notice that it implies that, for any non-negative integer p ,

$$\begin{aligned} \mathbb{E}[(\Delta_R^{(m)})^p] &= \sum_{i_1, \dots, i_p=1}^N a_{i_1} \dots a_{i_p} \mathbb{E} \left[\prod_{k=1}^p \Delta_{i_k}^{(m)} \right] \\ &\sim \sum_{i_1, \dots, i_p=1}^N a_{i_1} \dots a_{i_p} \mathbb{E} \left[\prod_{k=1}^p \xi_{i_k} \right] = \mathbb{E} \left[\left(\sum_{i=1}^N a_i \xi_i \right)^p \right]. \end{aligned}$$

Now since $\sum_{i=1}^N a_i \xi_i$ is a mean 0 real Gaussian with the same variance as $\Delta_R^{(m)}$, we see that the moments of $\frac{\Delta_R^{(m)}}{\sqrt{\text{Var } \Delta_R^{(m)}}}$ converge to the moments of the standard real Gaussian, which implies (25).

It remains to establish (29). We begin by re-formulating the right-hand side, and so we introduce some notation. Throughout this computation the integers p_1, \dots, p_N and m are considered to be fixed, and we often ignore the dependence of other parameters on them. We define $\hat{p}_i = p_1 + \dots + p_i$ for $1 \leq i \leq N$, and note that $P = \hat{p}_N$ which we will use interchangeably according to the context. We define a new sequence of random variables $(\tilde{\xi}_r)_{1 \leq r \leq P}$ by

$$\begin{aligned} \tilde{\xi}_1 &= \tilde{\xi}_2 = \dots = \tilde{\xi}_{p_1} = \xi_1, \\ \tilde{\xi}_{p_1+1} &= \tilde{\xi}_{p_1+2} = \dots = \tilde{\xi}_{p_2} = \xi_2, \\ &\vdots \\ \tilde{\xi}_{\hat{p}_{N-1}+1} &= \tilde{\xi}_{\hat{p}_{N-1}+2} = \dots = \tilde{\xi}_{\hat{p}_N} = \xi_N. \end{aligned}$$

A partition $\mathcal{P} = \bigsqcup_k \{r_k, s_k\}$ is a partition of the set $\{1, \dots, P\}$ into pairs $\{r_k, s_k\}$. We always label the partition so that $r_k < s_k$ and $r_k < r_{k'}$ for $k < k'$. Of course if P is odd then no such partition exists. Now [Jan97, Theorem 1.28] implies that

$$(30) \quad \mathbb{E} \left[\prod_{j=1}^N \xi_j^{p_j} \right] = \sum_{\mathcal{P}} \prod_k \mathbb{E}[\tilde{\xi}_{r_k} \tilde{\xi}_{s_k}].$$

In particular this expectation is zero if P is odd.

We now consider the left-hand side of (29). Since

$$\Delta_i^{(m)} = \int_{\Gamma_i} \frac{\partial}{\partial \hat{n}_i} \left(-\frac{\gamma}{2} + \sum_{\alpha=1}^m \frac{c_\alpha}{\alpha!} :|\hat{f}_R(z)|^{2\alpha}: \right) |dz_i| = \sum_{\alpha=1}^m \frac{c_\alpha}{\alpha!} \int_{\Gamma_i} \frac{\partial}{\partial \hat{n}_i} :|\hat{f}_R(z)|^{2\alpha}: |dz_i|$$

we see that, denoting by $\Gamma_i^{p_i}$ the Cartesian product of p_i copies of Γ_i ,

$$\mathbb{E} \left[\prod_{i=1}^N (\Delta_i^{(m)})^{p_i} \right] = \sum_{\alpha_1, \dots, \alpha_P=1}^N \prod_{r=1}^P \frac{c_{\alpha_r}}{\alpha_r!} \mathbb{E} \left[\int_{\prod_{i=1}^N \Gamma_i^{p_i}} \frac{\partial^P}{\partial \hat{n}_1 \dots \partial \hat{n}_P} \prod_{r=1}^P :|\hat{f}_R(z_r)|^{2\alpha_r}: \prod_{r=1}^P |dz_r| \right].$$

Lemma 9 allows us to apply Lemma 8 to see that

$$(31) \quad \mathbb{E} \left[\prod_{i=1}^N (\Delta_i^{(m)})^{p_i} \right] = \sum_{\alpha_1, \dots, \alpha_P=1}^m \prod_{r=1}^P \frac{c_{\alpha_r}}{\alpha_r!} \int_{\prod_{i=1}^N \Gamma_i^{p_i}} \frac{\partial^P}{\partial \hat{n}_1 \dots \partial \hat{n}_P} \mathbb{E} \left[\prod_{r=1}^P :|\hat{f}_R(z_r)|^{2\alpha_r}: \right] \prod_{r=1}^P |dz_r|.$$

We will compute the asymptotics of this expression via the diagram formula. For $1 \leq i, j \leq N$ and $1 \leq \alpha \leq m$ we write

$$\rho_{ij}(\alpha) = \int_{\Gamma_i} \int_{\Gamma_j} \frac{\partial^2}{\partial \hat{n}_i \partial \hat{n}_j} \left| \hat{K}_R(z_i, z_j) \right|^{2\alpha} |dz_j| |dz_i|.$$

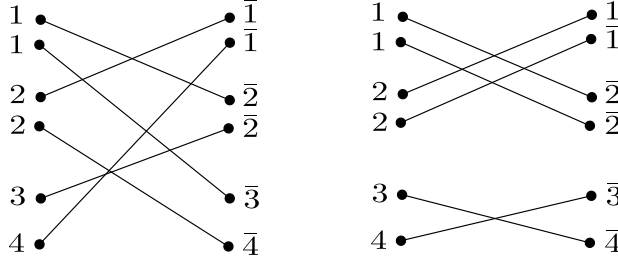


FIGURE 5. An irregular and a regular diagram for $P = 4$, $\alpha_1 = \alpha_2 = 2$ and $\alpha_3 = \alpha_4 = 1$.

We then have, from (28),

$$(32) \quad \text{Cov}(\Delta_i^{(m)}, \Delta_j^{(m)}) = \sum_{\alpha=1}^m c_{\alpha}^2 \rho_{ij}(\alpha).$$

3.3.1. *Diagrams.* Given non-negative integers $\alpha_1, \dots, \alpha_P$, a *diagram* \mathcal{D} is a graph with $2(\alpha_1 + \dots + \alpha_P)$ vertices such that:

- For each $1 \leq r \leq P$ there are α_r vertices labelled r and α_r vertices labelled \bar{r} .
- Each vertex has degree exactly 1.
- Each edge joins a vertex labelled r to a vertex labelled \bar{s} for $r \neq s$.

Note that there are choices of $\alpha_1, \dots, \alpha_P$ such that no such diagram exists, for example if $\alpha_1 > \alpha_2 + \dots + \alpha_P$. We denote the edges (respectively the vertices) of \mathcal{D} by $e(\mathcal{D})$ (respectively $v(\mathcal{D})$).

Recall that

$$\widehat{K}_R(z_r, z_s) = \frac{K_R(z_r, z_s)}{\sqrt{K_R(z_r, z_s)K_R(z_r, z_s)}} = \exp\{R^2(z_r \bar{z}_s - \tfrac{1}{2}|z_r|^2 - \tfrac{1}{2}|z_s|^2)\}.$$

The *value* of a diagram is

$$V(\mathcal{D}) = \prod_{(r, \bar{s}) \in e(\mathcal{D})} \widehat{K}_R(z_r, z_s).$$

The diagram formula [Jan97, Theorem 3.12] implies that

$$(33) \quad \mathbb{E}\left[\prod_{r=1}^P :|\widehat{f}_R(z_r)|^{2\alpha_r}:\right] = \sum_{\mathcal{D}} V(\mathcal{D}).$$

We say that a diagram is *regular* if the set $\{1, \dots, P\}$ can be partitioned into pairs $\{r_k, s_k\}$ such that each edge of the diagram is of the form (r_k, \bar{s}_k) or (s_k, \bar{r}_k) for some k ; otherwise the diagram is said to be *irregular*, see Figure 5. Note that if P is odd then all diagrams are irregular. We again label the partition so that $r_k < s_k$ and $r_k < r_{k'}$ for $k < k'$.

Combining (31) and (33) we have

$$(34) \quad \mathbb{E} \left[\prod_{i=1}^N (\Delta_i^{(m)})^{p_i} \right] = \sum_{\alpha_1, \dots, \alpha_P=1}^m \prod_{r=1}^P \frac{c_{\alpha_r}}{\alpha_r!} \sum_{\mathcal{D}} \int_{\prod_{r=1}^P \Gamma_i^{p_i}} \frac{\partial^P}{\partial \hat{n}_1 \dots \partial \hat{n}_P} V(\mathcal{D}) \prod_{r=1}^P |dz_r|.$$

We now split this sum into two pieces, by splitting $\sum_{\mathcal{D}} = \sum_{\text{regular } \mathcal{D}} + \sum_{\text{irregular } \mathcal{D}}$. We estimate each contribution separately - we shall see that the regular contribution will give us the main term on the right-hand side of (29) while the irregular contribution will give the error term. We begin by computing the regular part exactly.

The regular contribution.

We define the *multiplicity vector* $\vec{B} = (\beta_1, \dots, \beta_Q)$ of a regular diagram by $\beta_k = \alpha_{r_k} = \alpha_{s_k}$; here $P = 2Q$. Notice that for a regular diagram

$$V(\mathcal{D}) = \prod_k |\hat{K}_R(z_{r_k}, z_{s_k})|^{2\beta_k}.$$

Given a regular diagram \mathcal{D} with partition \mathcal{P} and multiplicity vector \vec{B} , we have

$$\begin{aligned} \int_{\prod_{r=1}^P \Gamma_i^{p_i}} \frac{\partial^P}{\partial \hat{n}_1 \dots \partial \hat{n}_P} V(\mathcal{D}) \prod_{r=1}^P |dz_r| &= \int_{\prod_{r=1}^P \Gamma_i^{p_i}} \frac{\partial^P}{\partial \hat{n}_1 \dots \partial \hat{n}_P} \prod_{k=1}^Q |\hat{K}_R(z_{r_k}, z_{s_k})|^{2\beta_k} \prod_{r=1}^P |dz_r| \\ &= \prod_{k=1}^Q \int_{\Gamma_{r_k}} \int_{\Gamma_{s_k}} \frac{\partial^2}{\partial \hat{n}_{r_k} \partial \hat{n}_{s_k}} |\hat{K}_R(z_{r_k}, z_{s_k})|^{2\beta_k} |dz_{s_k}| |dz_{r_k}| \\ &= \prod_{k=1}^Q \rho_{r_k, s_k}(\beta_k). \end{aligned}$$

We now need to count the number of regular diagrams with partition \mathcal{P} and multiplicity vector \vec{B} . The ordering of the partition we specified, combined with the multiplicity vector \vec{B} uniquely defines the values $\alpha_1, \dots, \alpha_P$. Given these values we may permute the α_r vertices labelled r , independently for each r , to get all of the regular diagrams corresponding to these values, \mathcal{P} and \vec{B} . There are $\prod_{r=1}^P \alpha_r!$ such permutations. Noting that $\prod_{r=1}^P c_{\alpha_r} = \prod_{k=1}^Q c_{\beta_k}^2$, we get that the

regular contribution is

$$\begin{aligned}
\sum_{\alpha_1, \dots, \alpha_P=1}^m \prod_{r=1}^P \frac{c_{\alpha_r}}{\alpha_r!} \sum_{\text{regular } \mathcal{D}} \prod_{k=1}^Q \rho_{r_k, s_k}(\beta_k) &= \sum_{\beta_1, \dots, \beta_Q=1}^m \prod_{k=1}^Q c_{\beta_k}^2 \sum_{\mathcal{P}} \prod_{k=1}^Q \rho_{r_k, s_k}(\beta_k) \\
&= \sum_{\mathcal{P}} \sum_{\beta_1, \dots, \beta_Q=1}^m \prod_{k=1}^Q c_{\beta_k}^2 \rho_{r_k, s_k}(\beta_k) \\
&= \sum_{\mathcal{P}} \prod_{k=1}^Q \left(\sum_{\beta=1}^m c_{\beta}^2 \rho_{r_k, s_k}(\beta) \right) \\
&\stackrel{(32)}{=} \sum_{\mathcal{P}} \prod_{k=1}^Q \text{Cov}(\Delta_{r_k}^{(m)}, \Delta_{s_k}^{(m)}) \\
&\stackrel{(30)}{=} \mathbb{E} \left[\prod_{j=1}^N \xi_i^{p_i} \right].
\end{aligned}$$

The irregular contribution.

It remains to see only that the irregular contribution is $o(R^{P/2})$. Further, from (34), we see that it is enough to bound

$$\int_{\prod_{i=1}^N \Gamma_i^{p_i}} \frac{\partial^P}{\partial \hat{n}_1 \dots \partial \hat{n}_P} V(\mathcal{D}) \prod_{r=1}^P |dz_r|$$

for each irregular diagram \mathcal{D} . Recalling that

$$V(\mathcal{D}) = \prod_{(r, \bar{s}) \in e(\mathcal{D})} \hat{K}_R(z_r, z_s) = \prod_{(r, \bar{s}) \in e(\mathcal{D})} \exp\{R^2(z_r \bar{z}_s - \frac{1}{2}|z_r|^2 - \frac{1}{2}|z_s|^2)\}$$

we have

$$\log V(\mathcal{D}) = R^2 \sum_{(r, \bar{s}) \in e(\mathcal{D})} (z_r \bar{z}_s - \frac{1}{2}|z_r|^2 - \frac{1}{2}|z_s|^2).$$

Now there are α_r edges in the sum of the form (r, \bar{s}) for some s , and α_r edges of the form (s, \bar{r}) for some s . We therefore have

$$\log V(\mathcal{D}) = R^2 \sum_{(r, \bar{s}) \in e(\mathcal{D})} (z_r(\bar{z}_s - \bar{z}_r)) = R^2 \sum_{(r, \bar{s}) \in e(\mathcal{D})} (\bar{z}_s(z_r - z_s))$$

from which we conclude that, for $1 \leq t \leq P$,

$$\frac{\partial}{\partial z_t} V(\mathcal{D}) = R^2 V(\mathcal{D}) \sum_{s: (t, \bar{s}) \in e(\mathcal{D})} (\bar{z}_s - \bar{z}_t)$$

and

$$\frac{\partial}{\partial \bar{z}_t} V(\mathcal{D}) = R^2 V(\mathcal{D}) \sum_{r: (r, \bar{t}) \in e(\mathcal{D})} (z_r - z_t).$$

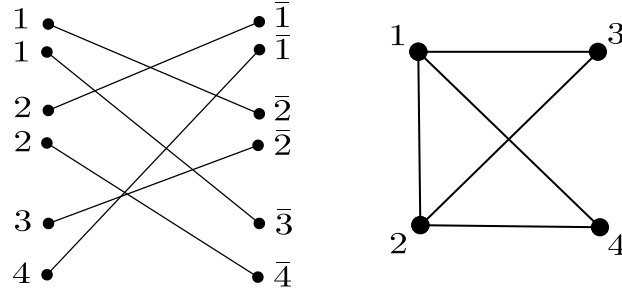


FIGURE 6. A diagram and its reduced diagram.

By iterating this argument we see that we may bound

$$\left| \frac{\partial^P}{\partial \hat{n}_1 \dots \partial \hat{n}_P} V(\mathcal{D}) \right|$$

by a (finite) linear combination of terms of the form

$$|V(\mathcal{D})| R^{2(P-\gamma)} \mathfrak{P}_{P-2\gamma},$$

where $0 \leq \gamma \leq \lfloor \frac{P}{2} \rfloor$ is an integer and $\mathfrak{P}_{P-2\gamma}$ is a product of $P - 2\gamma$ factors (not necessarily distinct) of the form $|z_r - z_s|$ with $(r, \bar{s}) \in e(\mathcal{D})$. To finish the proof it therefore suffices to see that

$$R^{2(P-\gamma)} \int_{\prod_{i=1}^N \Gamma_i^{p_i}} |V(\mathcal{D})| \mathfrak{P}_{P-2\gamma} \prod_{r=1}^P |dz_r| = o(R^{P/2})$$

for any choice of γ and $\mathfrak{P}_{P-2\gamma}$.

We now fix γ and $\mathfrak{P}_{P-2\gamma}$ and make a reduction to allow us to estimate this quantity. From the irregular diagram \mathcal{D} we form the reduced diagram \mathcal{D}^* (see Figure 6) with P vertices (labelled 1 to P) such that:

- For each $1 \leq r, s \leq P$ there is at most one edge (r, s) .
- $(r, s) \in e(\mathcal{D}^*)$ if $(r, \bar{s}) \in e(\mathcal{D})$ or $(s, \bar{r}) \in e(\mathcal{D})$.

In other words we form \mathcal{D}^* from \mathcal{D} by glueing together the $2\alpha_r$ vertices labelled r or \bar{r} for each r , and ignoring the multiplicity of the edges of the resultant diagram. We decompose

$$\mathcal{D}^* = \bigcup_{u=1}^n \mathcal{D}_u$$

into n connected components that contain a_u vertices and contribute ℓ_u factors to $\mathfrak{P}_{P-2\gamma}$. Notice that $n < \frac{P}{2}$ since \mathcal{D} is irregular, and that

$$\sum_{u=1}^n a_u = P \quad \text{and} \quad \sum_{u=1}^n \ell_u = P - 2\gamma.$$

Moreover, since

$$|V(\mathcal{D})| = \prod_{(r,\bar{s}) \in e(\mathcal{D})} \exp\{-\frac{R^2}{2}|z_r - z_s|^2\} \leq \prod_{(r,s) \in e(\mathcal{D}^*)} \exp\{-\frac{R^2}{2}|z_r - z_s|^2\}$$

we may factorise the expression we seek to bound as

$$\begin{aligned} R^{2(P-\gamma)} \int_{\prod_{i=1}^N \Gamma_i^{p_i}} |V(\mathcal{D})| \mathfrak{P}_{P-2\gamma} \prod_{r=1}^P |dz_r| \\ \leq R^{2(P-\gamma)} \prod_{u=1}^n \int_{\prod \tilde{\Gamma}_r} \prod_{(r,s) \in e(\mathcal{D}_u)} e^{-\frac{R^2}{2}|z_r - z_s|^2} \mathfrak{P}_{\ell_u} \prod_{r \in v(\mathcal{D}_u)} |dz_r| \end{aligned}$$

where, $\tilde{\Gamma}_r = \Gamma_i$ for some i , $\prod \tilde{\Gamma}_r = \prod_{r \in v(\mathcal{D}_u)} \tilde{\Gamma}_r$ and \mathfrak{P}_{ℓ_u} means a product of ℓ_u factors of the form $|z_r - z_s|$ with $(r, s) \in e(\mathcal{D}_u)$. We will show that

$$(35) \quad \int_{\prod \tilde{\Gamma}_r} \prod_{(r,s) \in e(\mathcal{D}_u)} e^{-\frac{R^2}{2}|z_r - z_s|^2} \mathfrak{P}_{\ell_u} \prod_{r \in v(\mathcal{D}_u)} |dz_r| \lesssim (\log R)^{\ell_u/2} R^{-(\ell_u + a_u - 1)}$$

which will yield

$$\begin{aligned} R^{2(P-\gamma)} \int_{\prod_{i=1}^N \Gamma_i^{p_i}} |V(\mathcal{D})| \mathfrak{P}_{P-2\gamma} \prod_{r=1}^P |dz_r| &\lesssim R^{2(P-\gamma)} (\log R)^{\sum_u \ell_u/2} R^{-\sum_u (\ell_u + a_u - 1)} \\ &= (\log R)^{\frac{P}{2}-\gamma} R^{2P-2\gamma-(P-2\gamma+P-n)} \\ &= (\log R)^{\frac{P}{2}-\gamma} R^n = o(R^{P/2}) \end{aligned}$$

as claimed, since $n < \frac{P}{2}$.

Proof of estimate (35).

It remains only to prove (35). We formulate it as follows: Let G be a connected graph with a vertices, let $\{\tilde{\Gamma}_1, \dots, \tilde{\Gamma}_a\} \subset \{\Gamma_1, \dots, \Gamma_N\}$ be a collection of curves (we allow repetition) and let \mathfrak{P}_ℓ be a product of ℓ factors of the form $|z_r - z_s|$ with $1 \leq r, s \leq a$. Then

$$(36) \quad \int_{\prod \tilde{\Gamma}_r} \prod_{(r,s) \in e(G)} e^{-\frac{R^2}{2}|z_r - z_s|^2} \mathfrak{P}_\ell \prod_{r=1}^a |dz_r| \lesssim (\log R)^{\ell/2} R^{-(\ell+a-1)}.$$

First note that since $e^{-\frac{R^2}{2}|z_r - z_s|^2} \leq 1$ we may delete some of the edges of G to form a tree. By re-labelling the vertices we may assume that deleting the vertices labelled $1, \dots, r$ yields a connected graph, for every r . We denote by $s(r+1)$ the vertex that is joined to $r+1$ in this reduced graph. See Figure 7. Note that

$$\prod_{(r,s) \in e(G)} e^{-\frac{R^2}{2}|z_r - z_s|^2} \leq \prod_{r=1}^{a-1} e^{-\frac{R^2}{2}|z_r - z_{s(r)}|^2}.$$

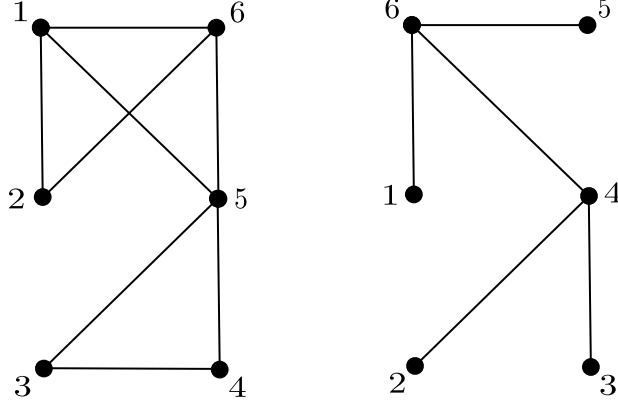


FIGURE 7. A connected graph and a tree formed by deleting some edges. The vertices are also re-labelled so that successively deleting the vertices labelled $1, 2, \dots$ yields a connected graph at every step, $s(1) = s(4) = s(5) = 6$ and $s(2) = s(3) = 4$.

Define $\varepsilon_R = \frac{\sqrt{(\ell+a-1)\log R}}{R}$ and note that if $|z_r - z_{s(r)}| > \varepsilon_R$ for some r then

$$\prod_{r=1}^{a-1} e^{-\frac{R^2}{2}|z_r - z_{s(r)}|^2} \leq R^{-(\ell+a-1)}.$$

Since \mathfrak{P}_ℓ is uniformly bounded on $\prod \tilde{\Gamma}_r$, and the curves have finite length, to show (36) it suffices to bound

$$\int_{\tilde{\Gamma}_a \times \prod_{r=1}^{a-1} C_r} \prod_{r=1}^{a-1} e^{-\frac{R^2}{2}|z_r - z_{s(r)}|^2} \mathfrak{P}_\ell \prod_{r=1}^a |dz_r|,$$

where $C_r = \{z_r \in \tilde{\Gamma}_r : |z_r - z_{s(r)}| \leq \varepsilon_R\}$. Note that in this new domain of integration we have $|z_r - z_s| \leq a\varepsilon_R$, which implies that $\mathfrak{P}_\ell \lesssim \varepsilon_R^\ell \lesssim \left(\frac{\sqrt{\log R}}{R}\right)^\ell$. It therefore suffices to see that

$$\int_{\tilde{\Gamma}_a \times \prod_{r=1}^{a-1} C_r} \prod_{r=1}^{a-1} e^{-\frac{R^2}{2}|z_r - z_{s(r)}|^2} \prod_{r=1}^a |dz_r| \lesssim R^{1-a}.$$

We claim that

$$(37) \quad \int_{C_1} e^{-\frac{R^2}{2}|z_1 - z_{s(1)}|^2} |dz_1| \lesssim R^{-1},$$

uniformly in the remaining variables. Applying this estimate $a-1$ times (replacing the index 1 by $2, \dots, a-1$) yields

$$\int_{\tilde{\Gamma}_a \times \prod_{r=1}^{a-1} C_r} \prod_{r=1}^{a-1} e^{-\frac{R^2}{2}|z_r - z_{s(r)}|^2} \prod_{r=1}^a |dz_r| \lesssim R^{1-a} \int_{\tilde{\Gamma}_a} |dz_a| \lesssim R^{1-a}.$$

Proof of estimate (37).

It remains to show (37). Fix $(z_2, \dots, z_a) \in \prod_{r=2}^{a-1} C_r \times \tilde{\Gamma}_a$ and define z_1^* to be the point in $\tilde{\Gamma}_1$ closest to $z_{s(1)}$. (If there are many points we choose one arbitrarily; it might be the case that $z_1^* = z_{s(1)}$.) We have $|z_1 - z_{s(1)}| \geq \frac{1}{2}|z_1 - z_1^*|$ since $|z_1 - z_{s(1)}| \geq |z_1 - z_1^*| - |z_1^* - z_{s(1)}| \geq |z_1 - z_1^*| - |z_1 - z_{s(1)}|$. This yields

$$\int_{C_1} e^{-\frac{R^2}{2}|z_1 - z_{s(1)}|^2} |dz_1| \leq \int_{C_1} e^{-\frac{R^2}{8}|z_1 - z_1^*|^2} |dz_1|,$$

which we bound exactly as in Section 2.1.1. Let $\gamma: [0, 1] \rightarrow \mathbb{C}$ be a parameterisation of $\tilde{\Gamma}_1$ satisfying $M_1 \leq |\dot{\gamma}(t)| \leq M_2$ and $M'_1|t - s| \leq |\gamma_j(t) - \gamma_j(s)| \leq M_2|t - s|$ with $M_1, M'_1 > 0$. Denote by t^* the (unique) value such that $\gamma(t^*) = z_1^*$. Notice that

$$C_1 \subseteq \left\{ z_1 = \gamma(t) : |t - t^*| \leq \frac{\varepsilon_R}{M'_1} \right\}.$$

We then bound

$$\begin{aligned} \int_{C_1} e^{-\frac{R^2}{8}|z_1 - z_1^*|^2} |dz_1| &\leq \int_{|t - t^*| \leq \frac{\varepsilon_R}{M'_1}} e^{-\frac{R^2}{8}|\gamma(t) - \gamma(t^*)|^2} |\dot{\gamma}(t)| dt \\ &\leq M_2 \int_{|t - t^*| \leq \frac{\varepsilon_R}{M'_1}} e^{-\frac{R^2}{8}(M'_1)^2(t - t^*)^2} dt \\ &= \frac{2\sqrt{2}M_2}{RM'_1} \int_{|s| \leq \frac{R\varepsilon_R}{2\sqrt{2}}} e^{-s^2} ds \\ &\lesssim R^{-1}. \end{aligned}$$

This completes the proof.

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